Rutgers - Physics Graduate Qualifying Exam
Quantum Mechanics: January 12, 2007

QA

Consider the wave function in one dimension

$$\psi(x) = C \exp -a|x| \tag{1}$$

where a is a positive real number. Normalize it, and then calculate $p^2\psi(x)$. Does your answer give the correct sign for $< p^2 >$?

QB

A particle of mass, m, is confined by the potential

$$\begin{aligned} V(x) &= \infty, \qquad & x < 0 \\ V(x) &= x^2, \qquad & x > 0. \end{aligned}$$

Find its energy eigenvalues.

QC1

Consider free particles

$$H_0 = \frac{p^2}{2m}$$

moving in 1-dimension of a ring of length L.

(a) Write down the energy levels and show each level $(except E = 0)$ is doubly degenerate.

(b) Briefly discuss perturbations from a degenerate level

(c) Add a perturbation

$$H_1 = -V_o e^{-x^2/a^2} \qquad a \ll L$$

Calculate the perturbed energy spectrum to first order and plot your results.

(d) Under what conditions is first order perturbation theory valid?

QC2

(a) Show that for the one-dimensional harmonic oscilator

$$\langle 0|e^{ikx}|0\rangle = e^{-k^2\langle 0|x^2|0\rangle/2}$$

(b) Calculate $\langle 0|\delta(x-a)|0\rangle$.

Interpret the result.

QD1

Consider the one dimensional particle of mass m in the potential $U(x)$ such that $U(x) \to 0$ as $|x| \to \infty$.

- (a) Write down the stationary Schroödinger equation in momentum representation.
- (b) Specialize the above mentioned equation for

$$U(x) = -\alpha \, \big(\delta(x-L) + \delta(x+L)\big) \; . \tag{2}$$

  Here $\delta(x)$ is the Dirac δ-function and α and L are some dimensionful positive constants.

- (c) Using the Schroödinger equation in momentum representation find the energy spectrum of bound states for (2) and the corresponding normalized wavefunctions.
- (d) The model (2) can be thought as a toy, one dimensional, model for the ion H_2^+, where the particle m plays a role of electron interacting with heavy "protons" located at $x = \pm L$. The interaction (2) induces an effective force acting between the "protons". Calculate this force in the limit of large inter-proton separation.

Useful integrals ($x \geq 0$)

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}t \; \frac{\cos(tx)}{1+t^2} &= \pi \; \mathrm{e}^{-x} \\
\int_{-\infty}^{\infty} \mathrm{d}t \; \frac{\cos^2(tx)}{(1+t^2)^2} &= \frac{\pi}{4} \, \Big(1 + (1+2x)\, \mathrm{e}^{-2x}\Big) \\
\int_{-\infty}^{\infty} \mathrm{d}t \; \frac{\sin^2(tx)}{(1+t^2)^2} &= \frac{\pi}{4} \, \Big(1 - (1+2x)\, \mathrm{e}^{-2x}\Big)
\end{aligned} \tag{3}$$

QD2

A fast neutron at energy 1 MeV collides with an unexcited Hydrogen atom. The atom was initially at rest. After the collision, the atom has been detected as moving at angle θ to the impact direction (see Fig.). Find a probability that the atom remained in the ground state.