Rutgers - Physics Graduate Qualifying Exam
Mechanics: January 10, 2007

MA

Suppose a block of mass m is projected with initial velocity, v_0, on a smooth horizontal plane, but there is air resistance proportional to v, i.e. $F(v) = -cv$. What is the maximum distance the block will move?

Solution MA

$$-cv = m\frac{dv}{dt}$$

$$t = \int_{v_0}^{v} -\frac{m}{c}\frac{dv}{v} = -\frac{m}{c}\ln\left(\frac{v}{v_0}\right)$$

So, $$v = v_0 e^{-ct/m}$$

$$x = \int_0^t v_0 e^{-ct/m}\, dt$$

$$= \frac{m}{c} v_0 \left(1 - e^{-ct/m}\right)$$

As $t \to \infty$, $x \to \frac{m v_0}{c}$

MB

(a) Show that the acceleration of a particle moving in a plane is given by:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \tag{1}$$

where r and θ are the usual plane polar coordinates.

(b) A particle moves on a spiral path such that its position is given by:

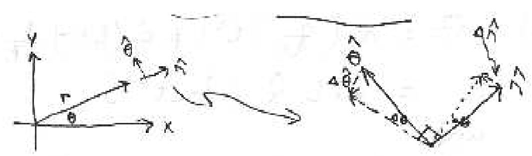
$$\begin{aligned} r &= bt^2 \\ \theta &= ct \end{aligned}$$

where b and c are constants. Find its velocity and acceleration.

(c) What is unusual about the radial component of the acceleration as t gets large?

Solution MB

a)



So, $\Delta\hat{r} \approx \Delta\theta\,\hat{\theta}$

and $\Delta\hat{\theta} \approx -\Delta\theta\,\hat{r}$

Then: $\vec{r} = r\hat{r}$

and $\vec{v} = \dot{r}\hat{r} + r\dfrac{d\hat{r}}{dt}$

$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$,

since $\dfrac{\Delta\hat{r}}{\Delta t} \approx \dfrac{\Delta\theta}{\Delta t}\hat{\theta}$

In a similar fashion,

$$\begin{aligned} \vec{a} &= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\ &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r}) \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \end{aligned}$$

b) $\vec{v} = \hat{r}\left(\frac{d}{dt}bt^2\right) + \hat{\theta}(bt^2)\frac{d}{dt}(ct)$

$= 2bt\,\hat{r} + bct^2\,\hat{\theta}$

and

$\vec{a} = \hat{r}(2b - bt^2c^2) + 4bct\,\hat{\theta}$

c) As "t" gets large, the radial component of the acceleration ~~go~~ becomes negative, yet "r" is a monotonically increasing function of time.

MC1

A planet has a mass, M, and radius R. A rocket of mass, m, is launched at an angle, θ, with respect to a radial vector from the center of the planet.

(a) Find the escape velocity for a vertical flight, $\theta = \deg 0$.

(b) Find the circular orbit speed for $r = R$, $\theta = \deg 90$.

(c) Find the maximum height, h (in units of R, i.e. h/R), measured from the center of the planet if the rocket is launched tangentially from the surface with a speed 3/4 the escape speed.

(d) At $r = h$, this rocket is to be put into a circular orbit. How much energy must be added or subtracted to do this?

SOLUTION TO ROCKET LAUNCH PROBLEM.

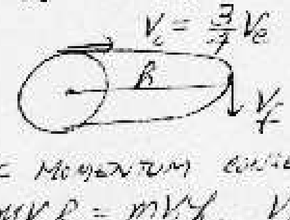
1. ESCAPE VELOCITY V_e

$$\frac{1}{2} m V_e^2 - \frac{GMm}{R} = 0 \qquad V_e = \sqrt{\frac{2GM}{R}}$$

2. ORBIT SPEED AT $r = R$

$$\frac{m V_0^2}{R} = \frac{GmM}{R^2} \qquad V_0 = \sqrt{\frac{GM}{R}}$$

(3).



$\measuredangle$ MOMENTUM CONSERVATION

$$m V_c R = m V_f r_f \qquad V_f = V_c \frac{R}{r_f} = \frac{3}{4} V_e \frac{R}{r_f}$$

CONSERVATION OF ENERGY

$$\frac{1}{2} m V_c^2 - \frac{GMm}{R} = \frac{1}{2} m V_f^2 - \frac{GMm}{r_f}$$

$$\frac{1}{2} \frac{9}{16} \frac{2GM}{R} - \frac{GM}{R} = \frac{1}{2}\left(\frac{R}{r_f}\right)^2 \frac{9}{16} \frac{2GM}{R} - \frac{GM}{R}\frac{R}{r_f}$$

$$\frac{9}{16} - 1 = \frac{9}{16}\left(\frac{R}{r_f}\right)^2 - \left(\frac{R}{r_f}\right) \qquad x = \frac{R}{r_f}$$

$$\frac{9}{16} x^2 - x + \frac{7}{16} = 0$$

$$(x-1)\left(\frac{9}{16} x - \frac{7}{16}\right) = 0 \qquad x = 1, \; x = \frac{7}{9} = \frac{R}{r_f} \quad r_f = \frac{9}{7} R \quad \frac{r_f}{R} = \frac{9}{7}$$

(4) $E_f = E_i = \frac{1}{2} m V_c^2 - \frac{GMm}{R} = \frac{1}{2} m \frac{9}{16} \frac{2GM}{R} - \frac{GMm}{R}$

$$= -\frac{7}{16} \frac{GMm}{R}$$

ORBIT AT $r = r_f$ $\qquad \frac{GmM}{r_f^2} = \frac{m V_f^2}{r_f} \qquad \frac{m V_f^2}{2} = + \frac{GMm}{2 r_f}$

$$E_{orb} = \frac{1}{2} m V_f^2 - \frac{GmM}{r_f} = -\frac{GmM}{2 r_f} = \frac{-GmM}{R} \frac{1}{2} \frac{R}{r_f}$$

$$= -\frac{1}{2} \frac{7}{9} \frac{GmM}{R} = \frac{-7}{18} \frac{GMm}{R}$$

$$E_{orb} - E_f = \left(-\frac{7}{18} + \frac{7}{16}\right) \frac{GMm}{R} = \frac{7}{2}\left(\frac{1}{8} - \frac{1}{9}\right) \frac{GMm}{R} = \frac{7}{2} \frac{1}{72} \frac{GMm}{R} > 0$$

MC2

Aerodynamic drag forces on an object of mass, m, are given

$$F_D = \frac{1}{2}\rho A C_D v^2, \tag{2}$$

where ρ is the density of air, A is the cross sectional area of the object, C_D is a drag coefficient and v is the speed of the object. Define a constant

$$K = \frac{\rho A C_D}{(2m)} \tag{3}$$

and express your answers in terms of this constant.

(a) A object is constrained to move horizontally subject only to F_D. The object has an initial speed, v_0 and position $x_0 = 0$.

- (i) Find the object's speed at time t.
- (ii) Find the object's position at time t.

(b) The same object is now released vertically from rest in a gravitational field with gravitational acceleration, g. Its initial position is $y_0 = 0$.

- (i) Find the object's speed at time, t, and find its terminal speed.
- (ii) Find the object's position as a function of t.

(c) Check your answers in a-i, a-ii, b-i and b-ii when $C_D = 0$.

Useful formulae:

$$\begin{aligned} \int (\tanh z) dz &= \ln(\cosh z) + c \\ \tanh z &= \{\ln(1+z) - \ln(1-z)\} \end{aligned}$$

SOLUTION TO AERODYNAMIC DRAG

MERIAM
SOLUTIONS TO CLASSICAL
MECHANICS

(1) $m\frac{d^2x}{dt^2} = -\frac{1}{2}\rho A C_D v^2 = m\frac{dv}{dt}$

$\frac{dv}{dt} = -\frac{1}{2}\frac{\rho A C_D}{m} v^2 = -kv^2. \qquad k = \frac{\rho A C_D}{2m}$

$\frac{dv}{v^2} = -k\,dt$

$\int_{V_0}^{V} \frac{dv}{v^2} = -k\int_0^t = -\frac{1}{V}\Big|_{V_0}^{V} = \frac{1}{V_0} - \frac{1}{V} = -kt \quad \frac{1}{V} = \frac{1}{V_0} + kt$

$V = \frac{V_0}{1+V_0 kt} \qquad C_D \to 0 \quad k \to 0 \quad V \to V_0$ ✓ (5) 1a

(2) $\frac{dx}{dt} = v = \frac{v_0}{1+v_0 kt} \qquad x = \int_0^t \frac{v_0\,dt}{1+kv_0 t} \quad \text{let } z = 1+kv_0 t$
$dz = kv_0\,dt$

$x = v_0 \frac{1}{kv_0}\int \frac{dz}{z} = \frac{1}{k}\ln z = \frac{1}{k}\ln(1+kv_0 t)$

$\xrightarrow[k\to 0]{} \frac{1}{k} k v_0 t = v_0 t$ ✓ (5) 1b

(3) $\frac{dv}{dt} = g - kv^2 \quad \text{TERMINAL } v \quad \frac{dv}{dt} = 0 \quad g = kv_t^2 \quad v_t = \sqrt{\frac{g}{k}}$

$\frac{dv}{g-kv^2} = dt = \frac{dv}{g(1-\frac{k}{g}v^2)} \qquad t = \int_0^V \frac{1}{g}\frac{dv}{1-\frac{k}{g}v^2} \quad z^2 = \frac{k}{g}v^2$

$t = \frac{1}{g}\sqrt{\frac{g}{k}}\int \frac{dz}{1-z^2} = \frac{1}{\sqrt{gk}}\int \frac{dz}{(1-z)(1+z)} = \frac{1}{\sqrt{gk}}\int \frac{1}{2}dz\left(\frac{1}{1+z} + \frac{1}{1-z}\right)$

$= \frac{1}{2\sqrt{gk}}\left(\ln\left(1+\sqrt{\frac{k}{g}}v\right) - \ln\left(1-\sqrt{\frac{k}{g}}v\right)\right) = \frac{1}{\sqrt{gk}}\,\text{arc tanh}\sqrt{\frac{k}{g}}v$

$\xrightarrow{k\to 0} \frac{1}{2\sqrt{gk}}\left(\sqrt{\frac{k}{g}}v - \left(-\sqrt{\frac{k}{g}}v\right)\right) = t = \frac{2v}{2g} \quad v = gt$ ✓

(4) $\frac{\sqrt{k}\,dx}{\sqrt{g}\,dt} = \tanh \sqrt{gk}\,t \quad \text{FROM} \quad t = \frac{1}{\sqrt{gk}}\,\text{arc tanh}\sqrt{\frac{k}{g}}\,v$

$v = \sqrt{\frac{g}{k}}\tanh\sqrt{gk}\,t = \frac{dx}{dt} \quad x = \sqrt{\frac{g}{k}}\int \tanh\sqrt{gk}\,t\,dt \quad z = \sqrt{gk}\,t$

$x = \sqrt{\frac{g}{k}}\frac{1}{\sqrt{gk}}\int \tanh z\,dz = \frac{1}{(\sqrt{k})^2}\ln\cosh\sqrt{gk}\,t \to \frac{1}{(\sqrt{k})^2}\ln\frac{e^{\sqrt{gk}t}+e^{-\sqrt{gk}t}}{2}$

$\to \frac{1}{k}\ln(1+gkt^2/2) \to \frac{1}{k}gkt^2/2 = \frac{1}{2}gt^2$ ✓

MD1

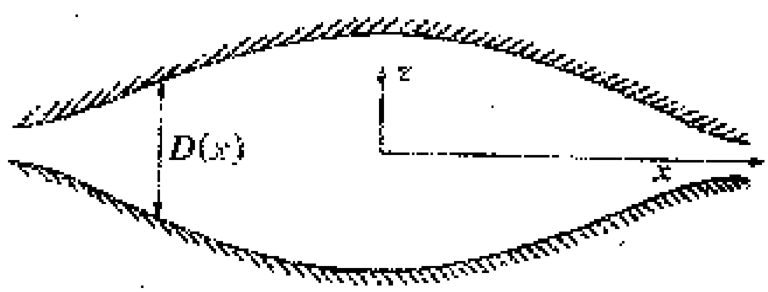
A particle of mass m moves freely between two symmetrical fixed surfaces, as shown below. It bounces off of both surfaces elastically, and the separation betwen the surfaces is given by

$$D(x) = D_0 \left[\frac{\cosh \alpha x}{\cosh 2\alpha x} \right],$$

where D_0 and α are constants.

(a) If $|\dot{z}| >> |\dot{x}|$ for the particle, then, in addition to the total energy, there is another approximate constant of the motion. Write an expression for it.

(b) If the particle starts at the origin in the xz-plane, moving with speed v_0 at a small angle β with respect to the z-axis, find an approximate expression for the value of x_{max}, defined to be the maximum distance the particle will reach in its motion in the x direction, and describe what happens after that.



Solution: MD1

(a) In a reference frame moving with the particle, it sees a very slowly changing effective Hamiltonian for $|\dot{z}| >> |\dot{x}|$. Thus, the action corresponding to the z motion is an **adiabatic invariant**.

$$J_z \equiv \oint p_z \, dz \equiv 2m|\dot{z}|D(x) \approx 2D_0 v_0 \cos\beta.$$

(b) From energy conservation,

$$\dot{x}^2 = v_0^2 - \dot{z}^2 = v_0^2 \left[1 - \frac{D_0^2}{D^2(x)} \cos^2 \beta \right],$$

which vanishes for

$$\cos\beta = \frac{\cosh \alpha x}{\cosh 2\alpha x},$$

or

$$1 - \frac{1}{2}\beta^2 \approx \frac{1 + \frac{1}{2}\alpha^2 x^2}{1 + 2\alpha^2 x^2} \approx 1 - \frac{3}{2}\alpha^2 x^2,$$

to give

$$x_{max} \approx \frac{1}{\sqrt{3}} \frac{\beta}{\alpha}.$$

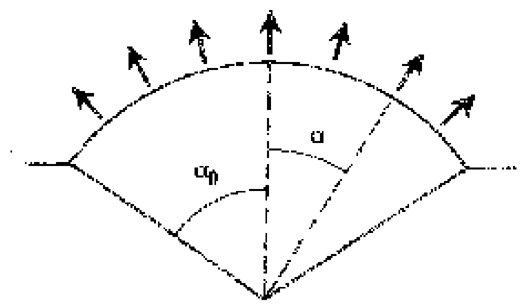
When it reaches x_{max}, the particle is reflected and moves to the left until it reaches $-x_{max}$. It oscillates between these points forever in the perfectly elastic limit. This behavior is exaclty analogous to the motion of a charged particle in a magnetic mirror (i.e. bottle) configuration.

MD2

Consider a lawn sprayer consisting of a spherical cap subtending an angle of $\alpha_0 = 45^o$ from the vertical, as shown in the picture below. The cap is provided with a large number of spherical holes through which water is ejected with the velocity v_0. The lawn is not uniformly sprayed if these holes are evenly spaced. How must

$$dN/d\alpha,$$

the number of holes per unit angle, be chosen to achieve uniform spraying of a circular area? Assume the radius of the sprinkling cap is very much less than the radius of the area to be sprayed, and the circumference of the cap is at the level of the lawn, as shown.



Solution: MD2

The range, b of an individual raindrop coming from a hole at angle α, is given by

$$b = \frac{v_0^2}{2g} \cos\alpha \sin\alpha.$$

The number of drops that exit the sprinkler through an infinitesimal solid angle, $d\Omega$, at angle α is proportional to

$$\frac{dN}{d\alpha} d\Omega \propto \frac{dN}{d\alpha} \sin\alpha \; d\alpha,$$

and, if the lawn is to be watered uniformly, the number of drops which land in an infinitesimal annular ring of radius b must be proportional to the area of that ring, so we must have

$$\frac{dN}{d\alpha} \propto \frac{b}{\sin\alpha} \frac{db}{da} \propto \cos\alpha(\cos^2\alpha - \sin^2\alpha).$$