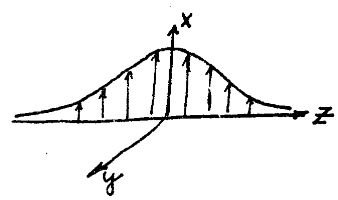
Rutgers - Physics Graduate Qualifying Exam
Electricity and Magnetism: January 10, 2007

EC1

A time-dependent, vacuum electromagnetic field in three dimensions, $(\hat{x}, \hat{y}, \hat{z})$ at time, $t = 0$, is shown in the figure.



It has the following form:

$$\vec{E}(\vec{r}, t = 0) = \hat{\imath} E_0 e^{-(z/a)^2}$$
$$\vec{H}(\vec{r}, t = 0) = 0$$

(a) Evaluate and sketch $\frac{\partial \vec{E}(\vec{r},t)}{\partial t}$ at $t = 0$.

(b) Evaluate and sketch $\frac{\partial \vec{H}(\vec{r},t)}{\partial t}$ at $t = 0$.

(c) Evaluate and sketch $\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$ at $t = 0$.

(d) What are the values of the fields $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ for a general time, t, satisfying the inequality, $(ct/a) \gg 1$?

(e) Sketch in a single diagram the fields found in (d).

(f) What is the physical significance of the electromagnetic fields appearing in this problem?

Solution: EC1

In a vacuum, with no charge or current, we work in a system of units where $\vec{B} = \vec{H}$ and $\vec{D} = \vec{E}$. Maxwell's equations can be written:

$$\vec{\nabla} \cdot \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{H} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{H}}{c \partial t}$$
$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{E}}{c \partial t}$$

In this problem, at $t = 0$,

$$\vec{E}(\vec{r}, t = 0) = \hat{\imath} E_0 e^{-(x/a)^2}$$
$$\vec{H}(\vec{r}, t = 0) = 0$$

(a) At $t = 0$,

$$\frac{\partial \vec{E}}{c\partial t} = \vec{\bigtriangledown} \times \vec{H} = 0 \tag{1}$$

(b) At $t = 0$,

$$\frac{\partial \vec{H}}{c\partial t} = -\vec{\bigtriangledown} \times \vec{E} = -\hat{j}\frac{\partial E_x}{\partial z} = \frac{2z}{a^2}E_0 e^{-(x/a)^2}\hat{j} \tag{2}$$

The drawing is in a coordinate system like that in the problem, with the vector derivative in the $y - z$ plane, passing through zero at $z = 0$, and anti-symmetric in z.

(c) At $t = 0$,

$$\frac{\partial^2 \vec{E}}{c^2\partial t^2} = \frac{\partial}{c\partial t}\left(\vec{\bigtriangledown} \times \vec{H}\right) = \vec{\bigtriangledown} \times \left(\frac{\partial \vec{H}}{c\partial t}\right) = \vec{\bigtriangledown} \times \left(\frac{2z}{a^2}E_0 e^{-(x/a)^2}\right)\hat{i} \tag{3}$$

(d) At $t = 0$, evaluate

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \hat{i}E_0\frac{\partial^2}{\partial z^2}e^{-(x/a)^2} = -\hat{i}E_0\frac{\partial}{\partial x}\left(\frac{2z}{a^2}E_0 e^{-(x/a)^2}\right) \tag{4}$$

This yields

$$\frac{\partial^2 \vec{E}}{\partial z^2} - \frac{\partial^2 \vec{E}}{c^2\partial t^2} = 0 \tag{5}$$

This can be true only if $\vec{E}$ is a sum of a function of $(z - ct)$ and another function of $(z + ct)$. The same is true of $\vec{H}$. Since $\vec{H} = 0$ at $t = 0$, its two functions have equal and opposite coefficients. The same coefficients, with like signs, apply to $\vec{E}$. This gives:

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \hat{i}\frac{E_0}{2}\left(e^{-((x-ct)/a)^2} + e^{-((x-ct)/a)^2}\right) \\ \vec{H}(\vec{r}, t) &= \hat{j}\frac{E_0}{2}\left(e^{-((x-ct)/a)^2} - e^{-((x-ct)/a)^2}\right) \end{aligned}$$

(e) The drawing should show two half-amplitude gaussian pulses showing E in the $x - z$ plane as above, one at $z \gg 0$ and the other at $z \ll 0$. Each should have a similar H pulses in the $y - z$ plane, the one at $z \gg 0$ is directed toward $+\hat{j}$ and the one at $z \ll 0$ is directed toward $-\hat{j}$.

(f) Two gaussian-shaped electromagnetic pulses, one travelling toward $z \to +\infty$ and the other toward $z \to -\infty$

EC2

Consider an iron sphere of constant magnetization, M.

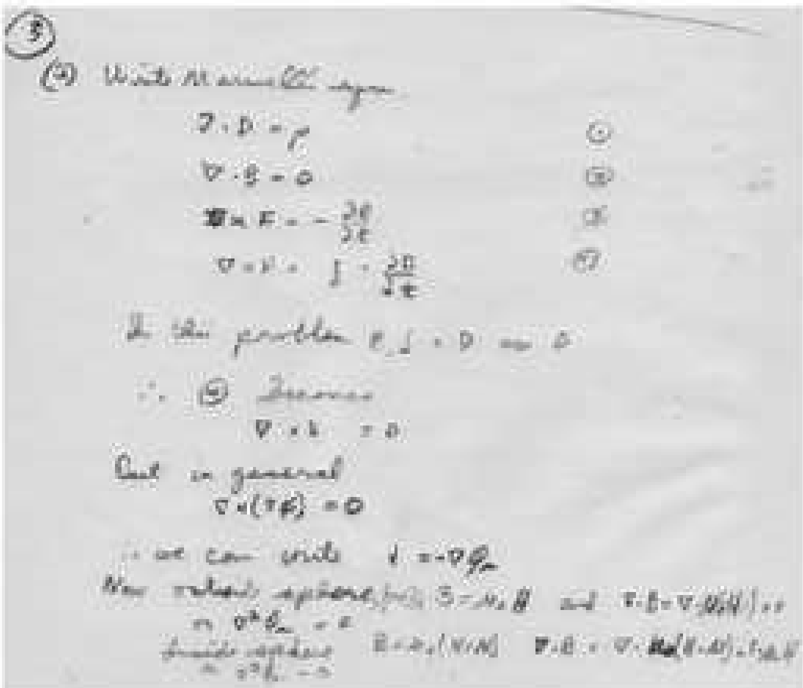
(a) Show that $\vec{H}$ can be written, $\vec{H} = -\vec{\nabla}\phi_m$, where ϕ_m is a scalar function satisfying $\nabla^2\phi_m = 0$.

(b) Derive the boundary conditions on ϕ_m.

(c) Find the potential both inside and outside the sphere.

(d) Make three sketches, quantitatively showing $\vec{B}$, $\vec{H}$ and $\vec{M}$.

E&M: Solution EC2, page 1



(3) (a) Write Maxwell's eqns

$\nabla \cdot D = \rho$ (i)

$\nabla \cdot B = 0$ (ii)

$\nabla \times E = -\frac{\partial B}{\partial t}$ (iii)

$\nabla \times H = J + \frac{\partial D}{\partial t}$ (iv)

In this problem E, J, D are 0

$\therefore$ (iv) becomes

$\nabla \times H = 0$

But in general

$\nabla \times (\nabla \phi) = 0$

$\therefore$ we can write $H = -\nabla \phi_m$

(b)

Using Gauss' theorem

$\therefore B_{\perp}$ is continuous

Using Stokes' Theorem

(3.3)

(c) Work in spher. Coord.

Let $\Phi_m = R(r)\,\Theta(\theta)\,\phi(\varphi)$

& use Laplace's eqn.

then

$$0 = \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\phi}\frac{d^2\phi}{d\varphi^2}$$

Assuming a radial θ dependence [illegible]

we find

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \text{const} = n(n+1)$$

General soln is

$$R = a\,r^n + \frac{b}{r^{n+1}}$$

Leaving

$$\frac{1}{\phi}\frac{d^2\phi}{d\varphi^2} = -n(n+1)\sin^2\theta - \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

again const [illegible]

but by symmetry $\phi = \text{const} = C$

and

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\sin\theta\,\Theta = 0$$

or, with $x = \cos\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + n(n+1)\Theta = 0$$

solutions are Legendre Polynomials $P_n(\cos\theta)$

(3.4)

Then, absorbing all constants into a_n & b_n

$$\Phi_m = \sum\left(a_n r^n + \frac{b_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Now bound cond at ∞ says $H = 0$ at ∞

$\therefore a_n = 0$ (as is arbitrary & constant [illegible])

$$\Phi_{m\,out} = \sum\frac{b_n}{r^{n+1}}P_n(\cos\theta)$$

Similarly at $r = 0$

$$\Phi_{m\,in} = \sum a_n r^n P_n(\cos\theta)$$

$$\vec\nabla\Phi_{out} = \hat e_r\sum\frac{-(n+1)b_n}{r^{n+2}}P_n(\cos\theta) + \frac{\hat e_\theta}{r}\sum\frac{b_n}{r^{n+1}}P_n'(\cos\theta)\sin\theta$$

$$\vec\nabla\Phi_{in} = \hat e_r\sum n a_n r^{n-1}P_n(\cos\theta) + \frac{\hat e_\theta}{r}\sum a_n r^n P_n'(\cos\theta)\sin\theta$$

Now $\vec n = \hat e_r$ at boundary of sphere

and $\vec M = \hat e_r\cos\theta\,|M| - \hat e_\theta\sin\theta\,|M|$ on surface

$$\vec n\cdot\left(\vec\nabla\Phi_{out} - \vec\nabla\Phi_{in}\right) = -\vec n\cdot\vec M$$

(B.C. #1)

at radius of sphere ρ

$$-\sum\frac{(n+1)b_n}{\rho^{n+2}}P_n(\cos\theta) - \sum n a_n\rho^{n-1}P_n(\cos\theta) = -|M|\cos\theta$$

or

$$\sum\left[\frac{(n+1)b_n}{\rho^{n+2}} + n a_n\rho^{n-1}\right]P_n(\cos\theta) = |M|\cos\theta$$

but because of orthogonality of $P_n(\cos\theta)$

$a_n = b_n = 0$ for $n \neq 1$ [illegible] following B.C.

E&M: Solution EC2, page 3

Using BC #2 (3.5)

$$\vec{n}\times\vec{\nabla}\varphi_{m\,in} = \vec{n}\times\nabla\varphi_{m\,out}$$

$$\cancel{\frac{1}{\rho}\sum a_n\rho^n P_n'(\cos\theta)\sin\theta} - \cancel{\frac{1}{\rho}\sum\frac{b_n}{r^{n+1}}P_n'(\cos\theta)\sin\theta} = 0$$

Thus we have, dropping subscript

$$\left(\frac{2b}{\rho^3}+a\right) = -M$$

$$a\rho - \frac{b}{\rho^2} = 0$$

$$a = +\frac{b}{\rho^3}$$

$$\frac{2b}{\rho^3}+\frac{b}{\rho^3} = -M$$

$$b = -\frac{M\rho^3}{3}$$

$$a = +\frac{M}{3}$$

Finally

$$\varphi_{m\,out} = \frac{-M\rho^3}{3r^2}\cos\theta$$

$$\varphi_{m\,in} = +\frac{Mr}{3}\cos\theta$$

(3.6)

With a little intuition we could have written immediately, recognizing symmetries

$$\varphi_{in} = ar\cos\theta \qquad \varphi_{out} = \frac{b}{r^2}\cos\theta$$

and solve as before

(d) Inside sphere

$$\vec{H}_{in} = -\vec{\nabla}\varphi_{in} = -\vec{\nabla}\frac{Mr\cos\theta}{3} = -\hat{e}_r\frac{M\cos\theta}{3} + \hat{e}_\theta\frac{M\sin\theta}{3}$$

Outside sphere

$$H_{out} = -\vec{\nabla}\varphi_{out} = +\hat{e}_r\frac{M\rho^3}{3r^3}\cos\theta - \hat{e}_\theta\frac{M\rho^3}{3r^3}\sin\theta$$

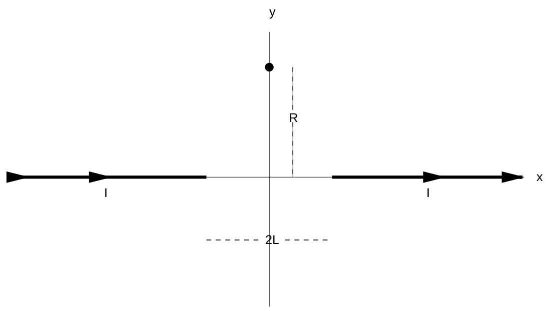
$\vec{M}$

$\vec{H}$

$\vec{B}$

ED1

A thin wire extends from $x = -\infty$ to $x = -L$ along the x axis, and another from $x = L$ to $x = \infty$, as shown in the figure below. A uniform current of magnitude I flows in the positive x direction in each wire. This means that positive charge is accumulating at a uniform rate at the end ($x = -L$) of the left wire, and negative charge is accumulating at the same rate at the end ($x = L$) of the right wire.



1. Use the circuital version of Maxwell's generalization of Ampere's law to calculate the magnetic field at the point $x = 0$, $y = R$, as designated by the solid dot in the figure.

2. Now use the Biot-Savart law to see what prediction it makes for this field. Is your answer the same as that for the previous part or different? Do you generally expect the Biot-Savart law to give the correct answer for this sort of magnetostatics problem where the displacement current plays a role?

3. Start from the general form of the Biot-Savart law

$$\vec{H}(\vec{r}_1) = \frac{1}{4\pi} \int \vec{j}(\vec{r}_2) \times \frac{\vec{r}_{12}}{|\vec{r}_{12}|^3} \, dV_2, \tag{6}$$

   where $\vec{H}$ is the magnetic field, $\vec{j}$ is the current density, and dV_2 is the volume element located at $\vec{r}_2$. Derive the most general form of the Ampere's Law Maxwell equation that follows from it. When is this derived equation valid?

Solution: ED1

1. Take the Ampere's law circuit to be a circle of radius R in the yz plane centered at the origin. There is no conduction current through the plane of this circle, only displacement current. We have[1]

$$\int \vec{H} \cdot d\vec{R} = \int \frac{\partial D_x}{\partial t} dA,$$

   where the line integral goes around the perimeter of the circle in the right-hand sense, and the area integral goes over the part of the yz plane enclosed by the circle. Letting r be the distance away from the x axis in the plane of the circle, we have

$$D_x(r) = 2 \cdot \frac{q}{4\pi(r^2 + L^2)} \cdot \frac{L}{\sqrt{r^2 + L^2}},$$

   where q is the instantaneous charge at the end of the left wire. The leading factor of 2 accounts for the equal effect of both wire ends, while the trailing fraction projects $\vec{D}$ in the direction perpendicular to the plane of the circle. Charge-current continuity requires

$$\frac{dq}{dt} = I.$$

[1]Even thought we are dealing with free space, we use D and H to avoid the presence of awkward dimensional constants depending on the unit choice. The rationalized form of Maxwell's Equations is used.

Then using symmetry, $\int \vec{H} \cdot d\vec{R} = 2\pi RH$, and $dA = 2\pi r dr$, we find

$$2\pi RH = I \int_0^R \frac{Lrdr}{(r^2 + L^2)^{3/2}}.$$

After evaluating the elementary integral, one finds

$$H = \frac{I}{2\pi R}\left[1 - \frac{L}{\sqrt{(R^2 + L^2)}}\right]. \tag{7}$$

$\vec{H}$ is in the z direction at the point indicated in the question. For large R the field modification due to the break in the wire would become unmeasurable, and the result then correctly goes to the one for a continuous wire, $H = I/2\pi R$.

2. Applying the Biot-Savart law, we have

$$H = 2 \cdot \int_{-\infty}^{-L} \frac{I\,dx}{4\pi(x^2 + R^2)} \cdot \frac{R}{\sqrt{(x^2 + R^2)}},$$

and evaluating the elementary integral, one obtains the same magnetic field as obtained by the Ampere Law method, Eq. (7) above. This result is expected. Although it is perhaps not obvious that the Biot-Savart law includes the displacement current contribution, for quasi-static problems like the above where the resulting magnetic field is constant, this contribution is in fact silently and automatically included. Showing this is the task of the final part of the problem.

3. Here to economize on complexity in equations with vector fields, we use numerical subscripts not to denote vector components, but to designate which position variable is involved. For example, we let $\vec{H}(\vec{r}_1) \equiv H_1$ and $\vec{j}(\vec{r}_2) \equiv j_2$. Subscripts on differential operators indicate which position variable is to be differentiated. In this notation we can write Eq. (6) as

$$H_1 = \int j_2 \times \frac{r_{12}}{4\pi|r_{12}|^3} dV_2 = \mathrm{curl}_1 \int \frac{j_2}{4\pi|r_{12}|} dV_2, \tag{8}$$

or

$$\mathrm{curl}_1\, H_1 = \mathrm{curl}_1\, \mathrm{curl}_1 \int \frac{j_2}{4\pi|r_{12}|} dV_2. \tag{9}$$

Now using the vector operator identity, $\mathrm{curl}\,\mathrm{curl} = \mathrm{grad}\,\mathrm{div} - \nabla^2$, to give

$$\mathrm{curl}_1\, H_1 = \mathrm{grad}_1 \int j_2 \cdot \mathrm{grad}_1 \frac{1}{4\pi|r_{12}|} dV_2 - \int j_2 \nabla_1^2 \frac{1}{4\pi|r_{12}|} dV_2. \tag{10}$$

The second term on the right is simply equal to j_1, since

$$\nabla_1^2 \frac{1}{4\pi|r_{12}|} = -\delta(r_{12}).$$

We can thus write

$$\mathrm{curl}\, H = j + j^L, \tag{11}$$

where

$$\begin{aligned} j_1^L &= \mathrm{grad}_1\, \mathrm{div}_1 \int \frac{j_2}{4\pi|r_{12}|} dV_2 && \text{(12a)} \\ &= \mathrm{grad}_1 \int j_2 \cdot \mathrm{grad}_1 \frac{1}{4\pi|r_{12}|} dV_2 && \text{(12b)} \\ &= \mathrm{grad}_1 \int j_2 \cdot \mathrm{grad}_2 \frac{(-1)}{4\pi|r_{12}|} dV_2 && \text{(12c)} \\ &= \mathrm{grad}_1 \int \frac{\mathrm{div}_2\, j_2}{4\pi|r_{12}|} dV_2, && \text{(12d)} \end{aligned}$$

where Eq. (12d) is obtained by integrating by parts. Now by charge–current continuity, we have

$$\operatorname{div} j = -\frac{\partial \rho}{\partial t}, \tag{13}$$

where ρ is the charge density. Using this in Eq. (12d) gives

$$j^L = \frac{\partial D^L}{dt}, \tag{14}$$

where

$$D_1^L = -\operatorname{grad}_1 \int \frac{\rho_2}{4\pi |r_{12}|} dV_2 \tag{15}$$

is simply the D-field that originates from the existent charge distributions. Thus we have

$$\operatorname{curl} H = j + \frac{\partial D^L}{\partial t} \tag{16}$$

and also from Eq. (9)

$$\operatorname{curl} H = 0. \tag{17}$$

These represent the fully correct version of the free-space Maxwell equations for magnetostatics, where H, j, and $\partial D/\partial t$ are time independent (or very slowly varying) and show that the Biot-Savart Law (Eq. (6)) implicitly includes the displacement current for this situation. $\partial D^L/\partial t$ is the *longitudinal* or *irrotational* component of the the displacement current density. For faster time variation, Faraday's law will normally produce a *transverse* or *solenoidal* component $\partial D^T/\partial t$, which would then have to be added to the right side of Eq. (16) to correctly describe the situation. However, for situations where the magnetic field is constant or slowly varying in time (quasistatic), the Biot-Savart Law (Eq. (6)) is always correct, while the form of Ampere's law without displacement current is not.

ED2

This problem deals with magnetic braking caused by eddy currents. Consider a *thin* conducting sheet of thickness l_z and conductivity σ lying in the xy plane, that moves in the y direction with a *slow* constant speed v. The sheet lies between the stationary pole pieces of a strong magnet, which produces a magnetic induction B in the $\hat{z}$ direction that is confined to a rectangle between the pole pieces. Said another way, the magnetic field has magnitude B within the rectangle and magnitude zero outside it. Take the rectangle to be centered at the origin, and to have side l_y in the y-direction and side l_x in the x-direction. Assume that the metal sheet has much larger linear dimensions than l_x and l_y. All questions below refer to quantities as measured in the laboratory frame in which the magnet pole pieces are stationary. Assume that the conducting sheet is non-magnetic (relative permeability of unity). Assume $l_z \ll l_x$ and $l_z \ll l_y$.

1. Taking $l_y \sim 1.5 l_x$ make a very rough sketch of the eddy current streamlines that would be expected. Note that outside the rectangle between the pole pieces, these streamlines must coincide with the lines of electric field. Do the eddy current streamlines also coincide with the electric field lines within the above rectangle? Why or why not?

2. Consider the case $l_x \gg l_y$. Calculate the braking force in terms of the given parameters and variables. Show that it is proportional to the area $l_x l_y$ of the pole pieces. Estimate the order of magnitude of this force for a sheet made of a typical conductor like copper or aluminum, when $B = 1$ T, $v = 10$ m/s, $l_x l_y = 1$ cm^2, $l_z = 1$ mm.

3. Consider now the opposite limit of $l_y \gg l_x$. Without trying to obtain this limit analytically, give arguments as to whether the braking force is still proportional to the area of the pole piece footprint, and if not, what the the dependence on this area is.

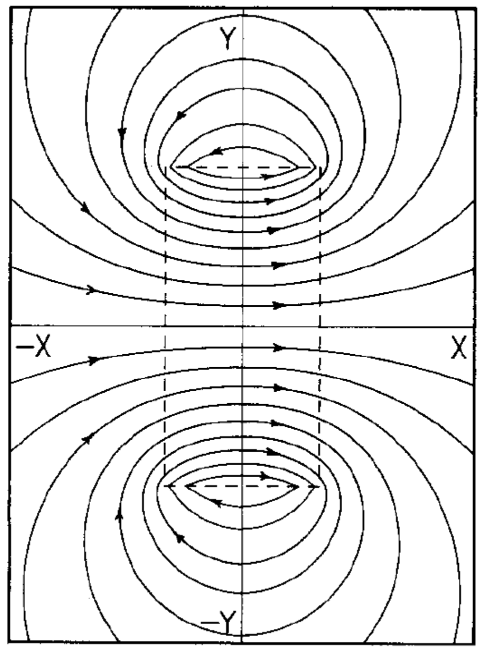
4. Now consider the case of arbitrary aspect ratio of the magnet footprint. Suppose you had calculated the exact electric field as a function of position. How would you use this information to calculate the current density everywhere and to calculate the the current density everywhere, and to calculate the magnetic braking force? What is called for here are quantitative and unambiguous mathematical expressions, not simply a discussion.

5. Calculate the electric field everywhere.

6. You were asked to answer the previous parts in the limit of a small velocity v. How small is "small"? Describe the essence of the approximation that is appropriate for the small v limit, and estimate the v at which it breaks down.

Solution: ED2

1. See the figure below. The current streamlines follow the lines of force, which within the rectangular footprint includes magnetic force as well as electric.



The figure here is based on the exact solution, and is taken from M. A. Heald, Am. J. Phys. **56**, 521 (1988). Of course your sketch needs only to have been crude and qualitative.

2. In this limit, the current flow becomes like that in a thin wire of cross section $A \equiv l_y l_z$ and length l_x and resistance $R = l_x/\sigma A$. The electric field enters the "wire" only near the ends, so that the force on the electrons inside of the pole piece footprint is essentially entirely magnetic, with an EMF $\mathcal{E} = vBl_x$ giving a current $I = \mathcal{E}/R = vBl_x\sigma A/l_x$. The braking force's magnitude is $F = IBl_x = v\sigma B^2 l_x l_y l_z$, and thus proportional to the volume of conductor between the pole pieces and to the area in the footprint. The current flows in the $\hat{x}$ direction, giving a braking force in the $-\hat{y}$ direction as expected. Typical resistivities are between 1 and 10 $\mu\Omega$cm giving forces between 10 and 100 N.

3. Here the magnetic force will be canceled by the electric force along most of the interior values of y giving currents within the footprint mostly near $y = \pm l_y/2$. Thus increasing l_y will no longer give any effect, and the braking force will no longer increase as the footprint area increases.

4. Letting E_x be the x component of the electric field within the thin sheet (or more accurately the z average of this quantity within the sheet), we have

$$\vec{F} = \int \vec{j} \times \vec{B}\, dV = l_z \sigma B \int_{-a}^{a} \int_{-b}^{b} (E_x + vB) dx\, dy\, (-\hat{y}), \tag{18}$$

where $a = l_x/w$ and $b = l_y/2$.

5. The current density $\vec{j}_B$ due to the $\vec{v} \times \vec{B}$ force terminates sharply at $x = \pm a$, implying that the current density due to the electric field $\vec{j}_E$ should cancel this sharp termination so that the total current density $\vec{j} = \vec{j}_B + \vec{j}_E$ satisfies $\operatorname{div} \vec{j} = 0$. Specifically, we have for $|y| < b$ and $|z| < l_z/2$

$$\operatorname{div} \vec{j}_B = \epsilon_0 \sigma v B[-\delta(x-a) + \delta(x+a)], \tag{19}$$

using SI units. This means that $\vec{j}_E$ should have the opposite divergence, and should have vanishing curl. Said another way, $\vec{E} = \vec{j}_E/\sigma$ is the field from the surface charge density $\pm\epsilon_0 vB$, on the rectangles of dimension $l_y \times l_z$ at $x = \pm a$. Because we are dealing with a conductor that cannot sustain an E-field that does not have a corresponding current density $\vec{j}_E = \sigma\vec{E}$, and because currents cannot flow in the $\hat{z}$ direction, the lines of $\vec{E}$ are constrained by the thin sheet to be in the xy plane. Therefore, according to Gauss's law, the electric field due to a short line segment of charge (length l_z in the z direction) with charge λ per unit length has a magnitude $\lambda/2\pi\epsilon_0 r$ at a distance r away from it in the xy plane, just the same as if the line segment were instead an infinite line. We can therefore calculate the electric field by simple integration,

$$\begin{aligned} E_x(x,y) &= \frac{vB}{2\pi}\int_{-b}^{b}\left[\frac{x-a}{(y-y')^2+(x-a)^2} - \frac{x+a}{(y-y')^2+(x+a)^2}\right]dy' \\ &= -\frac{vB}{2\pi}\left[\tan^{-1}\frac{b+y}{a-x} + \tan^{-1}\frac{b-y}{a-x} + \right. \\ &\qquad\qquad \left. \tan^{-1}\frac{b+y}{a+x} + \tan^{-1}\frac{b-y}{a+x}\right] \end{aligned} \tag{20}$$

and

$$\begin{aligned} E_y(x,y) &= \frac{vB}{2\pi}\int_{-b}^{b}\left[\frac{y-y'}{(y-y')^2+(x-a)^2} - \frac{y-y'}{(y-y')^2+(x+a)^2}\right]dy' \\ &= \frac{vB}{4\pi}\ln\frac{\left[(x-a)^2+(b+y)^2\right]\left[(x+a)^2+(b-y)^2\right]}{\left[(x+a)^2+(b+y)^2\right]\left[(x-a)^2+(b-y)^2\right]}. \end{aligned} \tag{21}$$

If you had difficulty with any of the first several parts of the problem, then Eqs. (18) and (20) could have been combined with appropriate series expansions to obtain those answers by brute force. In any case you would have been wise to cross check those more obvious limits.

6. The essence of the above approximation is that the magnetic field induced by the eddy currents is taken to be negligible in comparison with the applied field B. We can write $B_{\text{induced}} \sim \mu_0 I/l_y \sim \mu_0/l_y \cdot vB\sigma l_y l_z$. Thus the requirement $B_{\text{induced}}/B \ll 1$ implies that $v \ll (\mu_0\sigma l_z)^{-1}$. For the numbers of part 2, one obtains a velocity between 10 and 100 m/s.