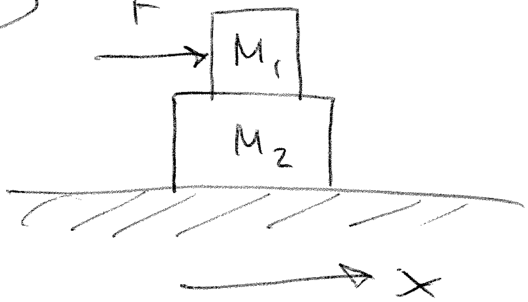
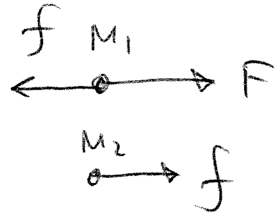
Classical Mech Solutions
Jan 11, 2006

MA1

(a)





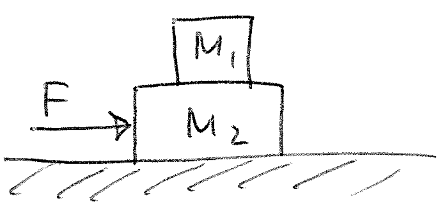
Let f be the friction force between the blocks.

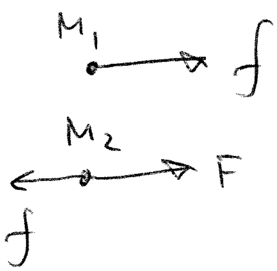
$$\begin{cases} F - f = M_1 a_1 \\ f = M_2 a_2 \end{cases} \qquad a_1 = a_2$$

Thus $F = f\left(1 + \frac{M_1}{M_2}\right)$ $\quad F_{max} = f_{max}\left(1 + \frac{M_1}{M_2}\right)$

$$f_{max} = \mu N = \mu M_1 g \implies \boxed{F_{max} = \mu M_1 g \left(1 + \frac{M_1}{M_2}\right)}$$

(b)





$$\begin{cases} f = M_1 a \\ F - f = M_2 a \end{cases}$$

$$\boxed{F_{max} = \mu M_1 g \left(1 + \frac{M_2}{M_1}\right)}$$

MB1

(a)

$$\left.\frac{du}{dr}\right|_{r_{min}} = 0 \qquad \frac{du}{dr} = \varepsilon\left(-\frac{12 r_0^{12}}{r^{13}} + \frac{12 r_0^{6}}{r^{7}}\right) = 0$$

$$\Rightarrow r_{min} = r_0$$

The depth is $u(r_{min}) = u(r_0) = -\varepsilon$

(b) Approximating $u(r)$ as quadratic potential at r_{min},

$$\omega = \sqrt{\left.\frac{d^2u}{dr^2}\right|_{r=r_0} \cdot \frac{1}{\mu}} \quad , \quad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m}{2}$$

We get $\left.\frac{d^2u}{dr^2}\right|_{r=r_0} = \frac{12\,\varepsilon \cdot 6}{r_0^2}$

$$\boxed{\omega = 12\sqrt{\frac{\varepsilon}{m r_0^2}}}$$

MC1

For motion in central field,

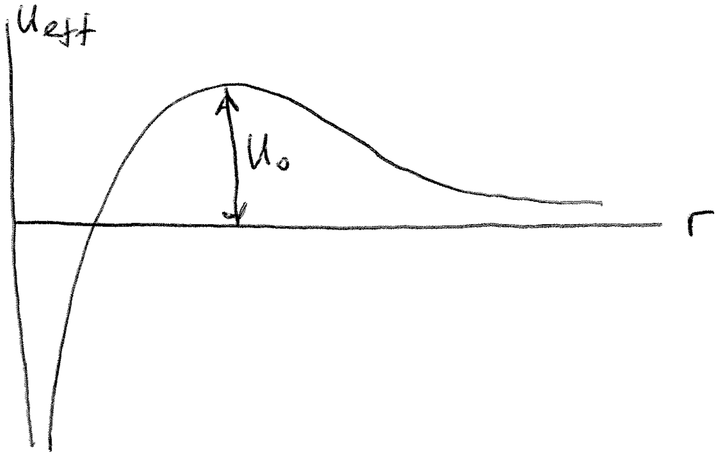
$$E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r), \quad \text{where}$$

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2mr^2}, \quad \text{and } \ell \text{ - is the}$$

conserved angular momentum of the particle.

For our case, $\ell = m v_0 \rho$

$$U_{eff}(r) = \frac{m \rho^2 v_0^2}{2r^2} - \frac{A}{r^n}$$



The maximum value of U_{eff} is $U_0 = \frac{1}{2}(n-2) A \left(\frac{m \rho^2 v_0^2}{A n} \right)^{\frac{n}{n-2}}$.

The particles which fall to the center are those for which $U_0 < E$. The condition $U_0 = E$ gives ρ_{max} $\Rightarrow$

$$\boxed{\sigma = \pi \rho_{max}^2 = \pi n (n-2)^{\frac{2-n}{n}} \left(\frac{A}{m v_0^2} \right)^{\frac{2}{n}}}$$

$$U = -\frac{GMm}{r} + 5.4\cdot 10^{-4}\,\frac{GMm}{r}\left(\frac{R_e}{r}\right)^2\left(3\cos^2\theta - 1\right) \equiv$$

$$-\frac{GMm}{r} + A\,\frac{3\cos^2\theta - 1}{r^3} \quad , \quad A \equiv 5.4\cdot 10^{-4}\, GMm\, R_e^2$$

In polar coordinates,

$$F_\theta = -\frac{1}{r}\frac{\partial U}{\partial \theta} = \frac{3A\cdot 2\cos\theta\cdot\sin\theta}{r^4} = \frac{3A\sin 2\theta}{r^4} \quad \therefore$$

$F_\theta \neq 0$ for $\theta \neq 0$, $\theta \neq \frac{\pi}{2}$.

For $\theta = 45^\circ$, $r = R_e$ $\quad F_\theta = \frac{3A}{R_e^4} = \frac{1.52\cdot 10^{-3}\, GMm}{R_e^2}$

$$\frac{F_\theta}{GMm/R_e^2} = 1.52\cdot 10^{-3} \quad \therefore$$

MDI The equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}=0$$

give
$$\begin{cases} \ddot{x}+\omega_0^2 x=\alpha y \\ \ddot{y}+\omega_0^2 y=\alpha x \end{cases} \qquad (1)$$

Look for solution $\begin{cases} x=A_x e^{i\omega t} \\ y=A_y e^{i\omega t} \end{cases}$, obtain from (1)

$$\begin{cases} A_x(\omega_0^2-\omega^2)=\alpha A_y \\ A_y(\omega_0^2-\omega^2)=\alpha A_x \end{cases} \qquad (2)$$

The characteristic equation

$$\begin{vmatrix} \omega_0^2-\omega^2 & -\alpha \\ -\alpha & \omega_0^2-\omega^2 \end{vmatrix}=0$$ gives

$$\boxed{\begin{aligned} \omega_1&=\sqrt{\omega_0^2-\alpha} \\ \omega_2&=\sqrt{\omega_0^2+\alpha} \end{aligned}}$$

$\therefore$

For $\omega = \omega_1$ equations (2) give $A_x = A_y$,
and for $\omega = \omega_2$, $A_x = -A_y$.

Hence, $x = \dfrac{Q_1 + Q_2}{\sqrt{2}}$ $\quad y = \dfrac{Q_1 - Q_2}{\sqrt{2}}$ where $Q_{1,2}$
are the normal coordinates.

Thus, $\boxed{Q_1 = \dfrac{x+y}{\sqrt{2}} \qquad Q_2 = \dfrac{x-y}{\sqrt{2}}}$ $\therefore$

Note: normalization coefficient $\frac{1}{\sqrt{2}}$ can be chosen differently, answers like $Q_1 = x+y$ etc are acceptable.

For $\alpha \ll \omega_0^2$, we have $\omega_{1,2} \simeq \omega_0 \mp \delta$
where $\delta = \frac{1}{2}\frac{\alpha}{\omega_0}$.

Thus, we have a sum of two oscillations with almost equal frequency, i.e. beats of frequency
$\boxed{\omega_2 - \omega_1 = \dfrac{\alpha}{\omega_0}}$ $\therefore$

e.g. $\cos(\omega_0 + \delta)t + \cos(\omega_0 - \delta)t = 2\cos(\omega_0 t)\cos(\delta t)$ etc.

In general, $x = \frac{1}{\sqrt{2}}\left(A e^{i\omega_0 t} e^{i\delta t} + B e^{i\omega_0 t} e^{-i\delta t}\right) = \frac{1}{\sqrt{2}} e^{i\omega_0 t}\left(A e^{i\delta t} + B e^{-i\delta t}\right)$
giving $A' \cos(\omega_0 t + \varphi_1) \cdot$ (slowly varying term of freq. δ)

For $0<t<T$, the equation of motion is

$$m\ddot{x} + m\omega^2 x = F(t) = \frac{F_0 t}{T} \qquad (1)$$

The solution is $x(t) = A_1 \cos \omega t + A_2 \sin \omega t + G(t)$

where $G(t)$ is any solution of (1).

Seek $G(t) = Bt$, gives $B = \frac{F_0}{Tm\omega^2}$.

Initial conditions then give

$$x(0) = 0 \quad \Rightarrow \quad A_1 = 0$$

$$\dot{x}(0) = 0 \quad \Rightarrow \quad A_2 = -\frac{F_0}{Tm\omega^3}$$

Thus, $x(t) = \frac{F_0}{mT\omega^3}(\omega t - \sin \omega t)$ for $0<t<T$. (2)

For $t>T$, we have an oscillator with a constant force applied. This will simply produce a displacement of the equilibrium position. Direct substitution into $m\ddot{x} + m\omega^2 x = F_0$ gives $\frac{F_0}{m\omega^2}$ for this displacement. Thus, for $t>T$

$$x(t) = C_1 \cos \omega(t-T) + C_2 \sin \omega(t-T) + \frac{F_0}{m\omega^2} \qquad (3)$$

Using (2) and (3), and the continuity condition of x and $\dot{x}$ at $t=T$ gives

$$C_1 = -\frac{F_0}{mT\omega^3}\sin\omega T \qquad C_2 = \frac{F_0}{mT\omega^3}\left(1-\cos\omega T\right).$$

The amplitude is $a = \sqrt{C_1^2 + C_2^2}$

$$\boxed{a = \frac{2F_0}{mT\omega^3}\sin\frac{\omega T}{2}}$$

As $T \to \infty$, $a \to 0$. In the limit of the adiabatic application of the force, the amplitude is zero ("the oscillator remains in the ground state" analogy from QM).