PROBLEM 1.

a.) First, we consider a unit length of cylinder to find the relationship between the total charge per unit length $\lambda$ and the charge density $\rho(r) = Ar$:

$$\lambda = \int Ar^2 dr d\phi = 2\pi A \int_0^R r^2 dr = 2\pi AR^3 / 3, \text{ or } A = \frac{3\lambda}{2\pi R^3}.$$ 

The total charge per unit length within radius $r$ ($r \leq R$) is then

$$Q(r) = 2\pi Ar^3 / 3 = \frac{\lambda r^3}{R^3}.$$ 

It is clear from symmetry that the field has only a radial component, which can be evaluated using a “Gaussian pillbox” and the integral form of Gauss’ law:

$$4\pi Q(r) = \oint E \cdot d\vec{S} = E_r \int r d\phi = 2\pi r E_r.$$ 

Inside the cylinder, $r \leq R$, one has $2\pi r E_r = 4\pi \lambda r^3 / R^3$ and $E_r = 2\lambda r^2 / R^3$.

Outside the cylinder, $r \geq R$ and $Q(r) = \lambda$, so $E_r = 2\lambda / r$.

The potential is derived from $E_r = -\frac{\partial}{\partial r} \Phi$. Choosing it to be 0 at $r = R$, we find:

Inside the cylinder, $\Phi(r) = -\frac{2\lambda}{3} \left( \frac{r^3}{R^3} - 1 \right).$

Outside the cylinder, $\Phi(r) = -2\lambda \ln \left( \frac{r}{R} \right).$

The figure is below.

![FIG. 1: The potential as a function of $r$.](image)
PROBLEM 2.

a.) To find the velocity of the c.m. system, we find a boost that makes the two photons have equal energies and equal but oppositely directed momenta. It is sufficient to use the first condition:

\[ E_{c.m.} = \gamma(1 - \beta)E_1 = \gamma(1 + \beta)E_2, \]

with \( E_1 \) the higher energy photon, and \( E_2 \) the lower energy photon. The boost goes in the same direction as the higher energy photon, 1, so that its c.m. energy is boosted lower. This leads to

\[ \beta = \frac{E_1 - E_2}{E_1 + E_2}. \]

We evaluate

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} = \frac{E_1 + E_2}{2\sqrt{E_1E_2}}. \]

Numerically, \( \beta \) is about \( 1 - 2 \times 10^{-12} \), and \( \gamma \approx 0.5 \times 10^6 \).

b) We apply the boost to photon 1:

\[ E_{c.m.} = \frac{E_1 + E_2}{2\sqrt{E_1E_2}}(1 - \frac{E_1 - E_2}{E_1 + E_2})E_1 = \frac{2E_2}{2\sqrt{E_1E_2}}E_1 = \sqrt{E_1E_2}. \]

Numerically, the c.m. photon energies are 1 keV.

PROBLEM 3A.

a.) To calculate \( \vec{B} \) in spherical coordinates, recalling \( \hat{r} \times \hat{\theta} = \hat{\phi} \), and that the vector potential has only a \( \hat{\phi} \) term, \( A_\phi = \frac{m\sin\theta}{r^2} \), we use the formula from the Jackson cheat sheet:

\[ \vec{B} = \nabla \times \vec{A} = \hat{r} \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta}(\sin \theta A_\phi) \right] + \hat{\theta} \left[ -\frac{1}{r} \frac{\partial}{\partial r}(rA_\phi) \right] \]

\[ \vec{B} = \frac{m}{r^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right] \]

Perhaps the simplest way to calculate the magnetic field in Cartesian coordinates is just to transform the spherical coordinate expression using \( \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \), and \( \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} + -\sin \theta \hat{z} \). We obtain

\[ \vec{B} = \frac{m}{r^3} \left[ 3 \sin \theta \cos \phi \hat{x} + 3 \sin \theta \cos \phi \hat{y} + (2 \cos^2 \theta - \sin^2 \theta) \hat{z} \right] \]

We now keep \( r^2 = x^2 + y^2 + z^2 \) and use \( \rho^2 = x^2 + y^2 \) for brevity, but replace \( \cos \theta \rightarrow \frac{z}{\rho} \), \( \sin \theta \rightarrow \frac{x}{\rho} \), \( \cos \phi \rightarrow \frac{\rho}{\rho} \), and \( \sin \phi \rightarrow \frac{\rho}{\rho} \) to obtain:

\[ \vec{B} = \frac{m}{\rho^3} \left[ 3xz \hat{x} + 3yz \hat{y} + (2z^2 - \rho^2) \hat{z} \right] \]

b.) The medium with \( \mu = \infty \) requires the field to be purely in the \( \hat{z} \) direction at the \( z = 0 \) plane, which is satisfied by an image dipole with the same magnitude and direction,
positioned at \( z = -d \). The B field for \( z > 0 \) is then the sum of the two dipole fields. We can obtain these most simply with the Cartesian coordinate \( \vec{B} \)-field formula, with the transformation \( z \rightarrow z \pm d \equiv z_{\pm} \). Also using \( r^2_{\pm} = x^2 + y^2 + (z \pm d)^2 \), the result is:

\[
\vec{B} = \frac{m}{r^3_{\pm}} \left( \frac{3x(z'_{\pm})}{r^5_{\pm}} + \frac{3x(z'_{\mp})}{r^5_{\mp}} \right) \hat{x} + \left( \frac{3y(z'_{\pm})}{r^5_{\pm}} + \frac{3y(z'_{\mp})}{r^5_{\mp}} \right) \hat{y} + \left( \frac{2(z'_{\pm})^2 - \rho^2}{r^5_{\pm}} + \frac{2(z'_{\mp})^2 - \rho^2}{r^5_{\mp}} \right) \hat{z}
\]

c.) The force exerted on the dipole is given by \( \vec{F} = \nabla (\vec{m} \cdot \vec{B}) \), where \( \vec{B} \) is the field from the image dipole, evaluated at \( x = y = 0, z = d \). Since \( \vec{m} \) is in the \( \hat{z} \) component of the field matters, and thus

\[
\vec{F} = \nabla \left( m^2 \frac{2(z + d)^2 - x^2 - y^2}{(x^2 + y^2 + (z + d)^2)^{5/2}} \right)
\]

It is clear from cylindrical symmetry that there is no \( x \) or \( y \) component to the force. In the algebra, this is because the \( x^2 \) and \( y^2 \) in the numerator and denominator lead to the derivatives being proportional to odd powers of \( x \) and \( y \) respectively, which vanish at \( x = y = 0 \). Thus, we only explicitly evaluate the derivative with respect to \( z \):

\[
\vec{F} = m^2 \left[ \frac{4(z + d)}{(x^2 + y^2 + (z + d)^2)^{5/2}} - \frac{(-5(z + d))(2(z + d)^2 - x^2 - y^2)}{(x^2 + y^2 + (z + d)^2)^{7/2}} \right] \hat{z}
\]

\[
\vec{F} = m^2 \frac{9(z + d)(x^2 + y^2) - 6(z + d)^3}{(x^2 + y^2 + (z + d)^2)^{7/2}} \hat{z} \rightarrow_x, y \rightarrow 0, z \rightarrow d \frac{-3m^2}{8d^4} \hat{z}
\]

Finally, as noted by WK in his caveat, a movement of the dipole also implies a movement of the image charge, which leads to a factor of 2 greater force:

\[
\vec{F} = \frac{-3m^2}{4d^4} \hat{z}
\]

The minus sign indicates that the force is attractive. And I hope this time I kept my factors of 2 straight...
PROBLEM 3B.

Assume the conducting plate is an \( xy \) plane at \( z = 0 \), and the electric dipole is \( \vec{p} = p_0 \hat{y} \) at \( z = d \) on the \( z \) axis.

a.) The electric dipole induces surface charges so that the electric field is purely normal to the \( xy \) plane at \( z = 0 \). This can be represented by an image electric dipole \( \vec{p}_{\text{image}} = -p_0 \hat{y} \), at position \( z = -d \). The vector potential for a free dipole \( \vec{p} \) at location \( \vec{d} \) is

\[
\vec{A}(\vec{r}) = -ik\vec{p} \frac{e^{ik|\vec{r}|d}}{|\vec{r} - d|}.
\]

Since vector potentials are additive, the total vector potential of the two dipoles is:

\[
\vec{A}(\vec{r}) = ik\vec{p} \left[ \frac{e^{ik|\vec{r}|d}}{|\vec{r} + d|} - \frac{e^{ik|\vec{r} - d|}}{|\vec{r} - d|} \right].
\]

Thus,

\[
\vec{A}(\vec{r}) = ik\frac{e^{ikr}}{r} \vec{p} \left[ \frac{e^{ikd\cos \theta}}{1 + (d/r) \cos \theta} - \frac{e^{-ikd\cos \theta}}{1 - (d/r) \cos \theta} \right],
\]

where we used the approximation \( d \ll r \) so \( |\vec{r} \pm d| \approx r \pm d \cos \theta \). We now expand, e.g., \( e^{ikd\cos \theta} = 1 + ikd \cos \theta + \ldots \) and \( (1 + (d/r) \cos \theta)^{-1} = 1 - (d/r) \cos \theta + \ldots \), and we take \( r \) to be large, to obtain

\[
\vec{A}(\vec{r}) \approx ik\frac{e^{ikr}}{r} \vec{p} [2ikd \cos \theta] \approx -2k^2d \cos \theta \frac{e^{ikr}}{r} \vec{p}.
\]

We find the \( \vec{B} \) field from \( \vec{B} = \nabla \times \vec{A} \). Since \( r \) is large, we neglect the \( \hat{\theta} \) and \( \hat{\phi} \) terms, which lead to extra powers of \( 1/r \). We also approximate \( \frac{\partial}{\partial r} \frac{e^{ikr}}{r} \approx ik \frac{e^{ikr}}{r} \), the leading order in \( r \). Thus,

\[
\vec{E}(\vec{r}) = \vec{B}(\vec{r}) \times \vec{n}.
\]

The Poynting vector

\[
\vec{S} = \frac{c}{8\pi} \vec{E} \times \vec{B}^* = \frac{c}{8\pi} |\vec{B}|^2 = \frac{c}{2\pi} k^6 d^2 \cos^2 \theta \ |\vec{n} \times \vec{p}|^2 / r^2.
\]

With the vectors written in terms of unit vectors

\[
\vec{p} = p \hat{e}_y, \quad \text{and} \quad \vec{n} = \sin \theta \cos \phi \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}_z,
\]

we find the radiation for the electric dipole located a distance \( d \) in front of a conducting plate:

\[
\frac{dP}{d\Omega} = r^2|\vec{S}| = \frac{c}{2\pi} k^6 d^2 \cos^2 \theta \ (\cos^2 \theta + \sin^2 \theta \cos^2 \phi).
\]

After integration over angles the total power is

\[
P = \frac{4}{15} c k^6 d^2 p^2.
\]

Compare this with the radiation pattern for a free electric dipole:

\[
\frac{dP}{d\Omega} = \frac{2\pi}{32\pi} k^4 c^3 p^2 \ (\cos^2 \theta + \sin^2 \theta \cos^2 \phi),
\]

and a total power of

\[
P = \frac{2\pi}{10\pi} k^4 c^3 p^2 \frac{4}{3}.
\]
The present problem differs in particular:
In the $y-z$ plane ($\cos \phi = 0$) the angular distribution of the radiation is as $\cos^4 \theta$, while for the free dipole it is like $\cos^2 \theta$.
In the $x-z$ plane ($\cos \phi = 1$) the angular distribution of the radiation is as $\cos^2 \theta$, while for the free dipole it is independent of $\theta$.

In general
\[
\frac{dP}{d\Omega} (\text{plate} + \text{dipole}) = 4k^2 d^2 \cos^2 \theta \times \frac{dP}{d\Omega} (\text{free dipole}).
\]

b.) If $\vec{p}$ is perpendicular to the conducting plate, image is $+\vec{p}$ at location $-\vec{d}$.

Because of the same direction of the two dipoles, the total vector potential is twice that of a free dipole. As a result the radiation pattern is multiplied by a factor 4, compared to the free electric dipole.

\[
\frac{dP}{d\Omega} (\text{plate} + \text{dipole}) = 4 \times \frac{dP}{d\Omega} (\text{free dipole}).
\]

**PROBLEM 4A.**
The total charge on the top plate is $Q = \sigma ab$.

a.) Students might recall the field in the capacitor is $\sigma / \varepsilon_0$. Otherwise, they use

\[
\vec{\nabla} \cdot \vec{E}_o = \frac{\rho}{\varepsilon_0} \rightarrow \vec{E}_o = -\frac{\sigma}{\varepsilon_0} \hat{z}.
\]

The Poynting vector is along the $+x$ axis:

\[
\vec{S} = \vec{E}_o \times \vec{H}_o = \frac{1}{\mu_0} \vec{E}_o \times \vec{B}_o = \frac{1}{\mu_0} \frac{-\sigma}{\varepsilon_0} \hat{z} \times B_0 \hat{y} = \frac{1}{\varepsilon_0 \mu_0} \sigma B_0 \hat{x} = c^2 \sigma B_0 \hat{x}.
\]

The momentum density is along $+x$ axis,

\[
\vec{p} = \frac{1}{c^2} \vec{E}_o \times \vec{H}_o = \sigma B_0 \hat{x},
\]

and the total momentum is then the volume integral

\[
\vec{P}_{\text{total}} = \int \vec{p} dV = \sigma B_0 abh \hat{x} = QB_0 h \hat{x}.
\]

b.) Now turn off $\vec{B}_o$ in time $\Delta t$. Since $\vec{B}_o = B_0 \hat{y}$, consider a closed loop in an $xz$ plane, at some fixed $y$, of the same area as a cross section through the capacitor, $A = ah$. We have an initial magnetic flux through the loop of $\int \vec{B}_o \cdot d\vec{S} = B_0 ah \hat{y}$. There is an induced $\vec{E}$ field

\[
\int \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B}_o \cdot d\vec{S} = \frac{-B_0 ah}{\Delta t}.
\]

With $h \ll a$, and the induced field opposing the decrease in the magnetic field, we see that

\[
\int \vec{E} \cdot d\vec{l} = 2a E_{\text{ind}} \rightarrow |E_{\text{ind}}| = \frac{B_0 h}{2\Delta t}
\]

with direction $+\hat{x}$ at the upper ($+\sigma$) plate, but $-\hat{x}$ at the lower ($-\sigma$) plate. The forces on both plates are then equal and in the same $+\hat{x}$ direction,

\[
\vec{F} = \frac{Q \vec{E}}{2\Delta t} \hat{x}.
\]
Integrating over time, \( \mathbf{p} = \int F \, dt \), gives a total momentum transfer to each plate of \( \mathbf{p} = (QB_0 h/2) \hat{i} \). Thus, in turning off the magnetic field, each plate receives half the momentum stored in the field.

**PROBLEM 4B.**

a.) The vector potential given describes an electromagnetic wave traveling along the +\( x \) axis. The \( \vec{E} \) and \( \vec{B} \) fields are calculated using

\[
\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\frac{1}{c} \frac{\partial f}{\partial t} \hat{y} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A} = \frac{\partial f}{\partial x} \hat{z}.
\]

Since the function \( f \) depends on \( u = x - ct \), we see that \( \frac{\partial f}{\partial t} = -c \frac{\partial f}{\partial x} \), so \( B_z = E_y \). The force on the electron is \( \vec{F} = q(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \), so

\[
F_x = -e \frac{1}{c} v_y B_z = -e \frac{1}{c} v_y E_y, \quad \quad F_y = -e E_y + \frac{1}{c} v_x B_z = -e E_y + e \frac{1}{c} v_x E_y, \quad \text{and} \quad \quad F_z = 0.
\]

Thus the electron moves in the \( xy \) plane. Recalling the hint above, we write the equations of motion as:

\[
\dot{v}_x = -\frac{e}{mc} v_y E_y, \quad \quad \dot{v}_y = -\frac{e}{m} E_y + \frac{e}{mc} v_x E_y.
\]

Assuming the light wave starts to interact with the electron at \( t = 0 \), the initial conditions are \( v_x(t = 0) = 0 \) and \( v_y(t = 0) = 0 \).

b.) Following the hint, we first evaluate

\[
\frac{d}{dt} \left( v_x^2 + v_y^2 \right) = 2v_x \dot{v}_x + 2v_y \dot{v}_y = 2v_x \frac{-e}{mc} v_y E_y + 2v_y \left( \frac{-e}{m} E_y + \frac{e}{mc} v_x E_y \right) = -\frac{2e}{m} v_y E_y = 2c \dot{v}_x.
\]

The trick was to substitute in the equations of motion, twice. One sees that \( v_x^2 + v_y^2 = 2cv_x \), and \( v_x \) thus must be positive. (This is the answer for part c.)

c.) Thus, the electron is pushed in the +\( x \) direction in the \( xy \) plane. The ultimate sign of the \( y \) component depends on the details of \( f(x - vt) \).

d.) We also see from the above relations that \( v_y \) and \( v_x \) are not independent, and we may eliminate one of them. The trick is to use \( v_y^2 = 2cv_x - v_x^2 = c^2 - (c - v_x)^2 \), \( c - v_x = \sqrt{c^2 - v_y^2} \) (note the useful integral) to eliminate \( v_x \) from the second \( (\dot{v}_y) \) equation of motion:

\[
\dot{v}_y = -\frac{e}{mc} E_y (c - v_x) \quad \rightarrow \quad \frac{\dot{v}_y}{\sqrt{c^2 - v_y^2}} = -\frac{e}{mc} \frac{E_y}{\sqrt{c^2 - v_y^2}} \frac{\partial f}{\partial t}.
\]

The integral of the left side over time is the useful integral:

\[
\int \frac{\dot{v}_y}{\sqrt{c^2 - v_y^2}} \, dt = \int \frac{1}{\sqrt{c^2 - v_y^2}} \, dv_y = \arcsin \left( \frac{v_y}{c} \right).
\]
The integral of the right side over time is

\[ \frac{e}{mc^2} \int \frac{\partial f}{\partial t} dt = \frac{e}{mc^2} (f(t) - f(-\infty)) = \frac{e}{mc^2} f(t). \]

Thus

\[ v_y(t) = c \sin \left( \frac{e}{mc^2} f(x - vt) \right), \]

and

\[ v_x = c - \sqrt{c^2 - v_y^2} = c - c \cos \left( \frac{e}{mc^2} f(x - vt) \right). \]

This seems to me to be a potentially difficult problem, much more so than 4A, since it requires several math tricks (several tricks → better technique?) to solve.