Qualifying Examination
Electricity and Magnetism Solutions
January 12, 2006

PROBLEM 1.

a.) First, we consider a unit length of cylinder to find the relationship between the total charge per unit length λ and the charge density $\rho(r) = Ar$:

$$\lambda = \int Ar^2 dr d\phi = 2\pi A \int_o^R r^2 dr = 2\pi AR^3/3, \text{ or } A = \frac{3\lambda}{2\pi R^3}.$$

The total charge per unit length within radius r ($r \leq R$) is then

$$Q(r) = 2\pi Ar^3/3 = \lambda r^3/R^3.$$

It is clear from symmetry that the field has only a radial component, which can be evaluated using a "Gaussian pillbox" and the integral form of Gauss' law:

$$4\pi Q(r) = \oint \vec{E}.d\vec{S} = E_r \int r d\phi = 2\pi r E_r.$$

Inside the cylinder, $r \leq R$, one has $2\pi r E_r = 4\pi\lambda r^3/R^3$ and $E_r = 2\lambda r^2/R^3$.
Outside the cylinder, $r \geq R$ and $Q(r) = \lambda$, so $E_r = 2\lambda/r$.

The potential is derived from $E_r = -\frac{\partial}{\partial r}\Phi$. Choosing it to be 0 at $r = R$, we find:

$$\text{Inside the cylinder, } \Phi(r) = -\frac{2\lambda}{3}\left(\frac{r^3}{R^3} - 1\right).$$

$$\text{Outside the cylinder, } \Phi(r) = -2\lambda \ln\left(\frac{r}{R}\right).$$

The figure is below.

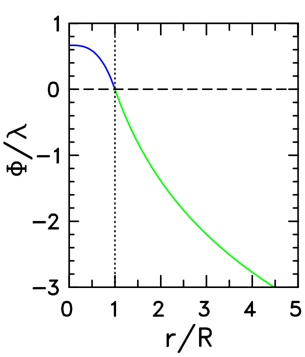

FIG. 1: The potential as a function of r.

PROBLEM 2.
a.) To find the velocity of the c.m. system, we find a
boost that makes the two photons have equal energies and equal but oppositely directed momenta. It is sufficient to use the first condition:

$$E_{c.m.} = \gamma(1-\beta)E_1 = \gamma(1+\beta)E_2,$$

with E_1 the higher energy photon, and E_2 the lower energy photon. The boost goes in the same direction as the higher energy photon, 1, so that its c.m. energy is boosted lower. This leads to

$$\beta = \frac{E_1 - E_2}{E_1 + E_2}.$$

We evaluate

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E_1 + E_2}{2\sqrt{E_1 E_2}}.$$

Numerically, β is about $1 - 2 \times 10^{-12}$, and $\gamma \approx 0.5 \times 10^6$.

b) We apply the boost to photon 1:

$$E_{c.m.} = \frac{E_1 + E_2}{2\sqrt{E_1 E_2}}(1 - \frac{E_1 - E_2}{E_1 + E_2})E_1 = \frac{2E_2}{2\sqrt{E_1 E_2}}E_1 = \sqrt{E_1 E_2}.$$

Numerically, the c.m. photon energies are 1 keV.

PROBLEM 3A.
a.) To calculate $\vec{B}$ in spherical coordinates, recalling $\hat{r} \times \hat{\theta} = \hat{\phi}$, and that the vector potential has only a $\hat{\phi}$ term, $A_\phi = \frac{m \sin\theta}{r^2}$, we use the formula from the Jackson cheat sheet:

$$\vec{B} = \nabla \times \vec{A} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right] + \hat{\theta}\left[-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$

$$\vec{B} = \frac{m}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

Perhaps the simplest way to calculate the magnetic field in Cartesian coordinates is just to transform the spherical coordinate expression using $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$, and $\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} + -\sin\theta\,\hat{z}$. We obtain

$$\vec{B} = \frac{m}{r^3}\left[3\sin\theta\cos\theta\cos\phi\,\hat{x} + 3\sin\theta\cos\theta\sin\phi\,\hat{y} + \left(2\cos^2\theta - \sin^2\theta\right)\,\hat{z}\right]$$

We now keep $r^2 = x^2 + y^2 + z^2$ and use $\rho^2 = x^2 + y^2$ for brevity, but replace $\cos\theta \to \frac{z}{r}$, $\sin\theta \to \frac{\rho}{r}$, $\cos\phi \to \frac{x}{\rho}$, and $\sin\phi \to \frac{y}{\rho}$ to obtain:

$$\vec{B} = \frac{m}{r^5}\left[3xz\,\hat{x} + 3yz\,\hat{y} + \left(2z^2 - \rho^2\right)\,\hat{z}\right]$$

b.) The medium with $\mu = \infty$ requires the field to be purely in the $\hat{z}$ direction at the $z = 0$ plane, which is satisfied by an image dipole with the same magnitude and direction, positioned at $z = -d$. The B field for $z > 0$ is then the sum of the two dipole fields. We can obtain these most simply with the Cartesian coordinate $\vec{B}$-field formula, with the transformation $z \to z \pm d \equiv z'_\pm$. Also using $r_\pm^2 = x^2 + y^2 + (z \pm d)^2$, the result is:

$$\frac{\vec{B}}{m} = \left(\frac{3x(z'_+)}{r_+^5} + \frac{3x(z'_-)}{r_-^5} \right) \hat{x} + \left(\frac{3y(z'_+)}{r_+^5} + \frac{3y(z'_-)}{r_-^5} \right) \hat{y} + \left(\frac{2(z'_+)^2 - \rho^2}{r_+^5} + \frac{2(z'_-)^2 - \rho^2}{r_-^5} \right) \hat{z}$$

c.) The force exerted on the dipole is given by $\vec{F} = \nabla(\vec{m} \cdot \vec{B})$, where $\vec{B}$ is the field from the image dipole, evaluated at $x = y = 0$, $z = d$. Since $\vec{m}$ is in the $\hat{z}$ direction, only the $\hat{z}$ component of the field matters, and thus

$$\vec{F} = \nabla \left(m^2 \frac{2(z+d)^2 - x^2 - y^2}{(x^2 + y^2 + (z+d)^2)^{5/2}} \right)$$

It is clear from cylindrical symmetry that there is no x or y component to the force. In the algebra, this is because the x^2 and y^2 in the numerator and denominator lead to the derivatives being proportional to odd powers of x and y respectively, which vanish at $x = y = 0$. Thus, we only explicitly evaluate the derivative with respect to z:

$$\vec{F} = m^2 \left[\frac{4(z+d)}{(x^2 + y^2 + (z+d)^2)^{5/2}} + \frac{(-5(z+d))(2(z+d)^2 - x^2 - y^2)}{(x^2 + y^2 + (z+d)^2)^{7/2}} \right] \hat{z}$$

$$\vec{F} = m^2 \frac{9(z+d)(x^2+y^2) - 6(z+d)^3}{(x^2 + y^2 + (z+d)^2)^{7/2}} \hat{z} \to_{x,y \to 0,\, z \to d} \frac{-3m^2}{8d^4} \hat{z}$$

Finally, as noted by WK in his caveat, a movement of the dipole also implies a movement of the image charge, which leads to a factor of 2 greater force:

$$\vec{F} = \frac{-3m^2}{4d^4} \hat{z}$$

The minus sign indicates that the force is attractive. *And I hope this time I kept my factors of 2 straight...*

PROBLEM 3B.

Assume the conducting plate is an xy plane at $z = 0$, and the electric dipole is $\vec{p} = p_0\hat{y}$ at $z = d$ on the z axis.

a.) The electric dipole induces surface charges so that the electric field is purely normal to the xy plane at $z = 0$. This can be represented by an image electric dipole $\vec{p}_{image} = -p_0\hat{y}$, at position $z = -d$. The vector potential for a free dipole $\vec{p}$ at location $\vec{d}$ is

$$\vec{A}(\vec{r}) = -ik\vec{p}\ \frac{e^{ik\cdot|\vec{r}-\vec{d}|}}{|\vec{r}-\vec{d}|}.$$

Since vector potentials are additive, the total vector potential of the two dipoles is:

$$\vec{A}(\vec{r}) = ik\,\vec{p}\left[\frac{e^{ik\cdot|\vec{r}+\vec{d}|}}{|\vec{r}+\vec{d}|} - \frac{e^{ik\cdot|\vec{r}-\vec{d}|}}{|\vec{r}-\vec{d}|}\right].$$

Thus,

$$\vec{A}(\vec{r}) = ik\frac{e^{ik\cdot r}}{r}\,\vec{p}\left[\frac{e^{ikd\cos\theta}}{1+(d/r)\cos\theta} - \frac{e^{-ikd\cos\theta}}{1-(d/r)\cos\theta}\right],$$

where we used the approximation $d \ll r$ so $|\vec{r} \pm \vec{d}| \approx r \pm d\ \cos\theta$. We now expand, e.g., $e^{ikd\cos\theta} = 1 + ikd\cos\theta + \ldots$ and $(1 + (d/r)\cos\theta)^{-1} = 1 - (d/r)\cos\theta\ldots$, and we take r to be large, to obtain

$$\vec{A}(\vec{r}) \approx ik\frac{e^{ikr}}{r}\,\vec{p}\,[2ikd\cos\theta] \approx -2k^2 d\cos\theta\frac{e^{ikr}}{r}\,\vec{p}$$

We find the $\vec{B}$ field from $\vec{B} = \nabla\times\vec{A}$. Since r is large, we neglect the $\hat{\theta}$ and $\hat{\phi}$ terms, which lead to extra powers of $1/r$. We also approximate $\frac{\partial}{\partial r}\frac{e^{ikr}}{r} \approx ik\frac{e^{ikr}}{r}$, the leading order in r. Thus,

$$\vec{B} \approx -2ik^3 d\cos\theta\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}$$

$\vec{E}(\vec{r}) = \vec{B}(\vec{r}) \times \vec{n}.$

The Poynting vector
$\vec{S} = \frac{c}{8\pi}\vec{E}\times\vec{B}^{\star} = \frac{c}{8\pi}|\vec{B}|^2 = \frac{c}{2\pi}k^6 d^2 cos^2\theta\ |\vec{n}\times\vec{p}|^2/r^2.$

With the vectors written in terms of unit vectors
$\vec{p} = p\ \vec{e}_y$, and
$\vec{n} = sin\theta\ cos\phi\ \vec{e}_x + sin\theta\ sin\phi\ \vec{e}_y + cos\theta\ \vec{e}_z,$
we find the radiation for the electric dipole located a distance d in front of a conducting plate:
$\frac{dP}{d\Omega} = r^2|\vec{S}| = \frac{c}{2\pi}k^6 d^2 cos^2\theta\ (cos^2\theta + sin^2\theta\ cos^2\phi).$

After integration over angles the total power is
$P = \frac{4}{15}c\ k^6 d^2 p^2.$

Compare this with the radiation pattern for a free electric dipole:
$\frac{dP}{d\Omega} = \frac{\mu_o}{32\pi}k^4 c^3 p^2\ (cos^2\theta + sin^2\theta\ cos^2\phi),$
and a total power of
$P = \frac{\mu_o}{16\pi}k^4 c^3 p^2 \frac{4}{3}.$

The present problem differs in particular:
In the $y-z$ plane ($cos\phi = 0$) the angular distribution of the radiation is as $cos^4\theta$, while for the free dipole it is like $cos^2\theta$.
In the $x-z$ plane ($cos\phi = 1$) the angular distribution of the radiation is as $cos^2\theta$, while for the free dipole it is independent of θ.

In general
$\frac{dP}{d\Omega}(plate + dipole) = 4k^2d^2\ cos^2\theta \times \frac{dP}{d\Omega}(free\ dipole)$.

b.) If $\vec{p}$ is perpendicular to the conducting plate, image is $+\vec{p}$ at location $-\vec{d}$.

Because of the same direction of the two dipoles, the total vector potential is twice that of a free dipole. As a result the radiation pattern is multiplied by a factor 4, compared to the free electric dipole.
$\frac{dP}{d\Omega}(plate + dipole) = 4 \times \frac{dP}{d\Omega}(free\ dipole)$.

PROBLEM 4A.

The total charge on the top plate is $Q = \sigma ab$.

a.) Students might recall the field in the capacitor is σ/ϵ_0. Otherwise, they use

$$\vec{\nabla} \cdot \vec{E_o} = \frac{\rho}{\epsilon_o} \quad \rightarrow \quad \vec{E_o} = -\frac{\sigma}{\epsilon_o}\hat{z}.$$

The Poynting vector is along the $+x$ axis:

$$\vec{S} = \vec{E_o} \times \vec{H_o} = \frac{1}{\mu_o}\vec{E_o} \times \vec{B_o} = \frac{1}{\mu_o}\frac{-\sigma}{\epsilon_o}\hat{z} \times B_o\hat{y} = \frac{1}{\epsilon_o\mu_o}\sigma B_o\hat{x} = c^2\sigma B_o\hat{x}.$$

The momentum density is along $+x$ axis,

$$\vec{p} = \frac{1}{c^2}\vec{E_o} \times \vec{H_o} = \sigma B_o\hat{x},$$

and the total momentum is then the volume integral

$$\vec{P}_{total} = \int \vec{p}dV = \sigma B_o abh\hat{x} = QB_o h\hat{x}.$$

b.) Now turn off $\vec{B_o}$ in time Δt. Since $\vec{B_o} = B_o\hat{y}$, consider a closed loop in an xz plane, at some fixed y, of the same area as a cross section through the capacitor, $A = ah$. We have an initial magnetic flux through the loop of $\int \vec{B_o} \cdot d\vec{S} = B_o ah\hat{y}$. There is an induced $\vec{E}$ field

$$\oint \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B_o}.d\vec{S} = \frac{-B_o ah}{\Delta t}.$$

With $h \ll a$, and the induced field opposing the decrease in the magnetic field, we see that

$$\oint \vec{E}.d\vec{l} = 2aE_{ind} \quad \rightarrow \quad |E_{ind}| = \frac{B_o h}{2\Delta t}$$

with direction $+\hat{x}$ at the upper ($+\sigma$) plate, but $-\hat{x}$ at the lower ($-\sigma$) plate. The forces on both plates are then equal and in the same $+\hat{x}$ direction,

$$\vec{F} = Q\vec{E} = \frac{QB_o h}{2\Delta t}\hat{x}.$$

Integrating over time, $\vec{p} = \int F dt$, gives a total momentum transfer to each plate of $\vec{p} = (QB_o h/2)\hat{x}$. Thus, in turning off the magnetic field, each plate receives half the momentum stored in the field.

PROBLEM 4B.

a.) The vector potential given describes an electromagnetic wave traveling along the $+x$ axis. The $\vec{E}$ and $\vec{B}$ fields are calculated using

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}\frac{\partial f}{\partial t}\hat{y} \text{ and } \vec{B} = \vec{\nabla} \times \vec{A} = \frac{\partial f}{\partial x}\hat{z}.$$

Since the function f depends on $u = x - ct$, we see that $\frac{\partial f}{\partial t} = -c\frac{\partial f}{\partial x}$, so $B_z = E_y$. The force on the electron is $\vec{F} = q(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B})$, so

$$F_x = -e\frac{1}{c}v_y B_z = -e\frac{1}{c}v_y E_y,$$

$$F_y = -eE_y + e\frac{1}{c}v_x B_z = -eE_y + e\frac{1}{c}v_x E_y, \text{ and}$$

$$F_z = 0.$$

Thus the electron moves in the xy plane. Recalling the hint above, we write the equations of motion as:

$$\dot{v}_x = -\frac{e}{mc}v_y E_y,$$

$$\dot{v}_y = -\frac{e}{m}E_y + \frac{e}{mc}v_x E_y.$$

Assuming the light wave starts to interact with the electron at $t = 0$, the initial conditions are $v_x(t = 0) = 0$ and $v_y(t = 0) = 0$.

b.) Following the hint, we first evaluate

$$\frac{d}{dt}\left(v_x^2 + v_y^2\right) = 2v_x\dot{v}_x + 2v_y\dot{v}_y = 2v_x\frac{-e}{mc}v_y E_y + 2v_y\left(\frac{-e}{m}E_y + \frac{e}{mc}v_x E_y\right) = \frac{-2e}{m}v_y E_y = 2c\dot{v}_x.$$

The trick was to substitute in the equations of motion, twice. One sees that $v_x^2 + v_y^2 = 2cv_x$, and v_x thus must be positive. (This is the answer for part c.)

c.) Thus, the electron is pushed in the $+x$ direction in the xy plane. The ultimate sign of the y component depends on the details of $f(x - vt)$.

d.) We also see from the above relations that v_y and v_x are not independent, and we may eliminate one of them. The trick is to use $v_y^2 = 2cv_x - v_x^2 = c^2 - (c - v_x)^2$, $c - v_x = \sqrt{c^2 - v_y^2}$ (note the useful integral) to eliminate v_x from the second ($\dot{v}_y$) equation of motion:

$$\dot{v}_y = -\frac{e}{mc}E_y\left(c - v_x\right) \quad \rightarrow \quad \frac{\dot{v}_y}{\sqrt{c^2 - v_y^2}} = -\frac{e}{mc}E_y = \frac{e}{mc^2}\frac{\partial f}{\partial t}.$$

The integral of the left side over time is the useful integral:

$$\int \frac{\dot{v}_y}{\sqrt{c^2 - v_y^2}}dt = \int \frac{1}{\sqrt{c^2 - v_y^2}}dv_y = \arcsin\left(\frac{v_y}{c}\right).$$

The integral of the right side over time is

$$\frac{e}{mc^2}\int \frac{\partial f}{\partial t}dt = \frac{e}{mc^2}(f(t) - f(-\infty)) = \frac{e}{mc^2}f(t).$$

Thus

$$v_y(t) = c\sin\left(\frac{e}{mc^2}f(x-vt)\right),$$

and

$$v_x = c - \sqrt{c^2 - v_y^2} = c - c\cos\left(\frac{e}{mc^2}f(x-vt)\right).$$

This seems to me to be a potentially difficult problem, much more so than 4A, since it requires several math tricks (several tricks → better technique?) to solve.