Rutgers - Physics Graduate Qualifying Exam
Quantum Mechanics: September 1, 2006

QA

J is an angular momentum vector with components J_x, J_y, and J_z. A quantum mechanical state is an eigenfunction of J^2 and J_z with eigenvalues $15/4\ \hbar^2$ and $1/2\hbar$ respectively.

1. Evaluate the expectation values $< J_x >$ and $< J_y >$ in this state.
2. Find the expectation values $< J_x^2 >$ and $< J_y^2 >$ in this state.

Show and justify any intermediate steps in your calculations.

QB

Consider the Schrödinger equation with a one-dimensional quartic potential: $V(x) = \lambda x^4$.

1. Find the lowest upper limit for the ground state energy of a particle of mass m using a Gaussian trial wave function of the form $\psi_t(x) = N \exp(-ax^2)$, where N is a normalizing constant and a is a parameter which can be varied.
2. Give a trial function which will give an upper limit for the energy of the first excited state in this potential. You do not need to carry out the calculation of the energy limit, but you should justify your choice of trial function.

You may find the following definite integrals useful:

$I_0 = \int_0^\infty exp(-px^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{p}}$

and, for positive integers n,

$I_n = \int_0^\infty x^{2n} exp(-px^2)dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n p^n} I_0$

QC1

The wave function of the bound state of a particle of mass m in the one-dimensional attractive delta-function potential $V(x) = -\lambda\delta(x)$ can be written as $\psi(x) = N exp(-a|x|)$, where N is the normalization constant.

1. Find a and the energy eigenvalue E in terms of λ and m.
2. Find the uncertainties in momentum and position Δp and Δx in terms of λ and m. Verify that the uncertainty relation is satisfied.

QC2

The eigenstates of the three-dimensional isotropic harmonic oscillator with potential $V(r) = \frac{1}{2}kr^2$ can be labelled either by the cartesian indices (n_x, n_y, n_z) or by the angular momentum indices (n, ℓ, m), with $n = n_x + n_y + n_z$.

1. What is the degeneracy of the nth energy level?
2. For $n = 0, 1$, and 2 list all of the degenerate states in both representations.
3. The wavefunctions in the angular momentum representation can be written $\psi_{n,\ell,m} = (u_{n,\ell}(r)/r)Y_{\ell,m}(\theta,\phi)$, where $u_{n,\ell}(r)$ is the radial wave function. Among the states in angular momentum representation you listed in (2) two have the same ℓ but different energies.
   Sketch the radial wave functions $u_{n,\ell}(r)$ for these two states, making clear how they differ.
4. Which of the $n = 1$ states in the cartesian representation listed in (2) is identical to a state in the angular momentum representation?
   Explain why.

QD1

The integral form of the Schrödinger equation for the scattering of a particle of mass m from a potential $V(\mathbf{r})$ is $\psi(\mathbf{r}) = exp(i\mathbf{k_i}\cdot\mathbf{r}) - (m/(2\pi\hbar^2)) \int \frac{exp(ik|\mathbf{r}-\mathbf{r'}|)}{|\mathbf{r}-\mathbf{r'}|} V(\mathbf{r'})\psi(\mathbf{r'})d^3r'$.

1. Use the large r limit of this expression to get an expression for the scattering amplitude f in terms of an integral involving the wave function ψ.
2. Use this expression to find the Rutherford cross section in the Born approximation, i.e find f to first order in V for $V = q_1q_2/(4\pi\epsilon_0 r)$ and use it to calculate the differential cross section. In the Born approximation $\psi(\mathbf{r'})$ is replaced by the initial plane wave, $f(\mathbf{k_f},\mathbf{k_i}) = -(m/(2\pi\hbar^2)) \int e^{-i\mathbf{q}\cdot\mathbf{r'}} V(\mathbf{r'})d^3r'$, where $\mathbf{q} = \mathbf{k_f} - \mathbf{k_i}$ is the momentum transfer.

QD2

The energy eigenvalues of the one-dimensional harmonic oscillator increase linearly with the quantum number n, while those for the one-dimensional infinite square well increase quadratically with n.

Consider a particle of mass m in the one-dimensional "V" potential, $V(x) = \lambda|x|$.

1. Use the Bohr-Sommerfeld condition or the WKB approximation to find the approximate dependence of the energy levels of this system on the quantum number n for large n. Give the estimates for the first four energy levels in terms of $\epsilon \equiv (\frac{\hbar^2\lambda^2}{2m})^{1/3}$.
2. Draw a graph showing the potential and the first four energy levels as horizontal lines. On each of these lines sketch the corresponding wave function. (You will probably want to make this graph as large as possible so that everything fits without overlap.)
3. Discuss the related problem (the quantum mechanics of the "bouncing ball") where $V(x) = \infty$ for $x < 0$, while $V(x) = \lambda x$ for $x > 0$. How are the energy levels and wave functions here related to those for the V potential?