August, 2006: Classical Mechanics - Solutions

MA

1. This is equivalent to two equal mass monkeys on opposite ends of the rope. Monday A climbs a distance d, the rope shortens a distance d, and all monkeys go up a distance $d/2$.

2. With both monkeys B and C climbing a distance d, they each go up a distance d relative to the pulley on the right; the pulley on the right descends a distance $d/2$, and all monkeys rise a distance $d/2$.

3. Relative to the pulley on the right, Monkey C climbs a distance d, the rope shortens a distance d, and both monkeys go up a distance $d/2$. Now since both monkeys go up a distance $d/2$, the pulley on the right descends a distance $d/4$, and all monkeys go up a distance $d/4$.

MB

1. Taking the positive direction to be to the right, the force acting along the rope are gravity on the block on the right, mg, a component of gravity acting on the block on the left, $-Mg\sin\theta$, and the frictional force acting on the block on the left, $\pm\mu Mg\cos\theta$. The blocks do not move to the right as long as $mg < Mg\sin\theta + \mu Mg\cos\theta$ or

$$\sin\theta + \mu\cos\theta > m/M.$$

   The blocks do not move to the left as long as $Mg\sin\theta < mg + \mu Mg\cos\theta$, or

$$\sin\theta - \mu\cos\theta < m/M.$$

2. The block to the right now has forces acting on it of gravity of mg downward, the normal force of the surface ma which provides the acceleration, and the frictional force which is up to μma. The block on the left has a normal (to the surface) force of $Mg\cos\theta$ as before, plus a component $-Ma\sin\theta$ from the acceleration. The frictional force is up to $\mu(Mg\cos\theta - Ma\sin\theta)$. Besides the frictional force, the force along the rope (to the left) is $Mg\sin\theta + Ma\cos\theta$. Thus, the condition for the blocks not to slide to the right is

$$mg < \mu ma + Mg\sin\theta + Ma\cos\theta + \mu(Mg\cos\theta - Ma\sin\theta)$$

   The condition for the blocks not to slide to the left is

$$Mg\sin\theta + Ma\cos\theta < mg + \mu ma + \mu(Mg\cos\theta - Ma\sin\theta).$$

MC1

1. For a mass on a spring, we know that $F = ma = -kx$ has solutions of the form $x = A\cos\omega t$, so that $ma = -m\omega^2 x$, and $\omega = \sqrt{k/m}$.

2. Assume the length of the spring is r_0, and it is stretched a distance r to provide the centripetal acceleration so the mass moves along a circular trajectory, or radius $r = r_0 + x$. Then $kx = mv^2/r$, or $x = mv^2/kr$. The angular momentum is $L_z = mvr$, so $x = mv^2/kr = L_z^2/kmr^3$.

3. The Lagrangian is $K = T - U$, with $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$, and $U = \frac{1}{2}k(r-r_0)^2$. In the θ coordinate, the equations of motion are derived from $\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$. This becomes $mr^2\ddot{\theta} = 0$, which integrated gives $mr^2\dot{\theta} = constant = L_z$. In the r coordinate, we have $m\ddot{r} - mr\dot{\theta}^2 + k(r-r_0) = 0$. ¿From the θ equations of motion, we replace $\dot{\theta}^2 = \frac{L_z^2}{m^2r^4}$ to obtain

$$m\ddot{r} - \frac{L_z^2}{mr^3} + k(r - r_0) = 0$$

.

Using $r = r_0 + x$, and $r^{-3} = (r_0 + x)^{-3} = r_0^{-3}(1 - x/r_0)^{-3} = r_0^{-3}(1 - 3x/r_0 \ldots)$, we obtain

$$m\ddot{x} + kx - \frac{L_z^2}{mr_0^3}(1 - 3x/r_0 \ldots)$$

$$\ddot{x} + \left(\frac{k}{m} + 3\frac{L_z^2}{m^2r_0^4}\right)x - \frac{L_z^2}{m^2r_0^3}\ldots = 0.$$

You can see that this will have solutions of the form $x = A\cos\omega t + B$, with ω picked so that

$$\omega^2 = \left(\frac{k}{m} + 3\frac{L_z^2}{m^2r_0^4}\right),$$

and with B picked so that $\omega^2 B - \frac{L_z^2}{m^2r_0^3} = 0$.

MC2

1. The frequency of a pendulum swing is well known. The Lagrangian is $L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$. The equation of motion is found from $\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$, which gives $ml^2\ddot{\theta} + mgl\sin\theta = 0$. Assuming θ is small, $\sin\theta \approx \theta\ldots$, and $\ddot{\theta} + (g/l)\theta = 0$, which is solved by $\theta = A\cos\omega t$, with $\omega = \sqrt{g/l}$.

2. The rod supplies a vertical force mg that counteracts gravity, and a horizontal force $mg\tan\theta$ that provides the centripetal acceleration of the mass. Since L_z is constant ($\dot{\phi}$ depends on θ), we write the centripetal force as $F_c = mr\dot{\phi}^2 = L_z^2/mr^3 = L_z^2/ml^3\sin^3\theta$. Then $mg\tan\theta = L_z^2/ml^3\sin^3\theta$ and $\frac{\sin^4\theta}{\cos\theta} = \frac{L_z^2}{m^2gl^3}$.
As $L_z \to 0$, $\theta \to 0$, while as $L_z \to \infty$, $\theta \to \pi/2$.

3. The Lagrangian is
$$L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\dot{\phi}^2 + mgl\cos\theta.$$
The ϕ equation of motion is $ml^2\sin^2\theta\ddot{\phi} = 0$, which integrates to $ml^2\sin^2\theta\dot{\phi} = \mathit{constant} \equiv L_z$. The θ equation of motion is $ml^2\ddot{\theta} - ml^2\sin\theta\cos\theta\dot{\phi}^2 + mgl\sin\theta = 0$. We replace $\dot{\phi} \to L_z/ml^2\sin^2\theta$ to obtain $ml^2\ddot{\theta} - \cos\theta L_z^2/ml^2\sin^3\theta + mgl\sin\theta = 0$, or
$$\ddot{\theta} - (\cos\theta L_z^2/m^2l^4\sin^3\theta) + (g/l)\sin\theta = 0.$$
It is no longer possible to assume that θ is small, but we may assume that if $\theta = \theta_0 + \delta$, with θ_0 the equilibrium angle from part 2), then δ is small. So $\sin\theta = \sin(\theta_0 + \delta) = \sin\theta_0\cos\delta + \cos\theta_0\sin\delta \approx \sin\theta_0 + \delta\cos\theta_0 + \ldots$ and $\cos\theta = \cos(\theta_0 + \delta) = \cos\theta_0\cos\delta - \sin\theta_0\sin\delta \approx \cos\theta_0 - \delta\sin\theta_0 + \ldots$. The equation of motion becomes
$$\ddot{\delta} - (\cos\theta_0 - \delta\sin\theta_0)L_z^2/m^2l^4(\sin\theta_0 + \delta\cos\theta_0)^3 + (g/l)(\sin\theta_0 + \delta\cos\theta_0) = 0.$$
The angular variables in the middle term are simplified using $(\cos\theta_0 - \delta\sin\theta_0)/(\sin\theta_0 + \delta\cos\theta_0)^3 = \cos\theta_0(1-\delta\tan\theta_0)\sin^{-3}\theta(1+\delta\cot\theta_0)^{-3} \approx \cos\theta_0\sin^{-3}\theta(1-\delta\tan\theta_0)(1-3\delta\cot\theta_0\ldots) \approx \cos\theta_0\sin^{-3}\theta(1-\delta(\tan\theta_0 + 3\cot\theta_0) + \ldots)$. We finally obtain
$$\ddot{\delta} + \left(\frac{g\cos\theta_0}{l} + \frac{L_z^2}{m^2l^4}\frac{\cos\theta_0}{\sin^3\theta_0}(\tan\theta_0 + 3\cot\theta_0)\right)\delta + \frac{g\sin\theta_0}{l} + \frac{L_z^2}{m^2l^4}\frac{\cos\theta_0}{\sin^3\theta_0} = 0.$$
This can be seen from inspection to have small oscillations, with solutions of the form $\delta = A\cos\omega t + B$. Here ω^2 can be identified with the coefficient of the δ term in the equation above, while B is determined from $\omega^2 B + \frac{g\sin\theta_0}{l} + \frac{L_z^2}{m^2l^4}\frac{\cos\theta_0}{\sin^3\theta_0} = 0$. You can see that in the limit $L_z \to 0$, $\cos\theta_0 \to 1$, $\sin\theta_0 \to 0$, and the usual planar pendulum solution is recovered – since L_z scales like $\sin^2\theta$, there is no divergence: $\ddot{\delta} + \frac{g}{l}\delta = 0$.

MD1

1. Since we have three blocks in one dimensional motion there are two normal modes. There is an anti-symmetric mode in which the two smaller blocks oscillate out of phase relative to each other, so that the larger block does not move. There is a symmetric mode in which the two smaller blocks oscillate in phase with each other, with the larger block oscillating out of phase so that there is no center of mass motion.

2. We write the Lagrangian for the system as

$$L = T - U = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}+\dot{x}_1)^2 + \frac{1}{2}m(\dot{x}+\dot{x}_2)^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2.$$

There are three equations of motion given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0.$$

These are:

$$\begin{aligned} x: &\quad M\ddot{x} + 2m\ddot{x} + m\ddot{x}_1 + m\ddot{x}_2 = 0, \\ x_1: &\quad m\ddot{x} + m\ddot{x}_1 + kx_1 = 0, \text{ and} \\ x_2: &\quad m\ddot{x} + m\ddot{x}_2 + kx_2 = 0. \end{aligned}$$

We solve for the anti-symmetric mode by subtracting the x_2 e.o.m. from the x_1 e.o.m.:

$$x_1 - x_2: \quad m\ddot{x}_1 - m\ddot{x}_2 + kx_1 - kx_2 = 0,$$

which, using $u = x_1 - x_2$, gives $m\ddot{u} + ku = 0$. This it the equation of a simple harmonic oscillator, with solution $u = A_-\cos(\omega_- t + \phi_-)$, where A_- and ϕ_- are chosen to satisfy initial conditions, and $\omega_-^2 = k/m$.

For the symmetric mode, add the x_1 and x_2 equations to obtain:

$$x_1 + x_2: \quad 2m\ddot{x} + m\ddot{x}_1 + m\ddot{x}_2 + kx_1 + kx_2 = 0.$$

Rewrite the x equation as:

$$x: \quad \ddot{x} = \frac{-1}{M+2m}(m\ddot{x}_1 + m\ddot{x}_2),$$

and insert in the $x_1 + x_2$ equation to obtain:

$$\left[\frac{-2m}{M+2m} + 1\right] m(\ddot{x}_1 + \ddot{x}_2) + k(x_1 + x_2) = 0.$$

Using $v = x_1 + x_2$, this simplifies to

$$\left[\frac{Mm}{M+2m}\right]\ddot{v} + kv = 0.$$

This it the equation of a simple harmonic oscillator, with solution $v = A_+\cos(\omega_+ t + \phi_+)$, where A_+ and ϕ_+ are chosen to satisfy initial conditions, and $\omega_+^2 = (k/m)(1 + 2m/M)$: you can see in the limit that $m/M \to 0$, we have two oscillators in phase at frequency $\omega^2 = k/m$.

3. The solutions for x_1 and x_2 are straightforward from the definitions of u and v. We have $x_1 = (u+v)/2 = (A_-/2)\cos(\omega_- t + \phi_-) + (A_+/2)\cos(\omega_+ t + \phi_+)$ and $x_2 = (u-v)/2 = (A_-/2)\cos(\omega_- t + \phi_-) - (A_+/2)\cos(\omega_+ t + \phi_+)$. For x, there is no motion related to the antisymmetric (-) mode, but there is possible constant c.m. motion, plus possible offsets. Thus $x = A_x\cos(\omega_+ t + \phi_+) + v_0 t + x_{off}$, where the constant A_x has to be chosen so that in pure symmetric motion the center of mass does not move: $MA_x + 2mA_+ = 0$, so $A_x = -2mA_+/M$.

MD2

1. First consider the forces on the small masses. The gravitational force, mg has a tangential component $mg\sin\theta$ and a radial component $mg\cos\theta$. The total radial force must be mv^2/R, since the small mass is moving in a circular path. The velocity v is given by energy conservation. With $\frac{1}{2}mv^2 = mgh$, $v = \sqrt{2gh} = \sqrt{2gR(1-\cos\theta)}$. Thus, the force exerted by the big ring on each small ring is the centripetal force minus the radial component of gravity:

$$F_{rr} = mv^2/R - mg\cos\theta = 2mg(1-\cos\theta) - mg\cos\theta = 2mg - 3mg\cos\theta.$$

   The horizontal forces on the big ring will cancel, and the total vertical force on the big ring becomes

$$F_{bv} = 2(2mg - 3mg\cos\theta)\cos\theta - Mg,$$

   and the ring moves up if this force is greater than 0. Note the extra factor of 2, from the two small rings. This solution still depends on θ, and we need to eliminate it. We do this by evaluating the derivative to find the maximum of the function:

$$\frac{dF_{bv}}{d\cos\theta} = 2(2mg - 6mg\cos\theta),$$

   which is 0 at $\cos\theta = 1/3$. The second derivative is clearly negative, so this is the desired maximum. Thus,

$$F_{bv}^{max} = 2(2mg - mg)\frac{1}{3} - Mg = \frac{2}{3}mg - Mg,$$

   and the limiting condition is

$$\frac{m}{M} > \frac{3}{2}.$$

2. For $M = 0$, we have

$$F_{bv} = 2(2mg - 3mg\cos\theta)\cos\theta,$$

   which becomes positive when $\cos\theta = 2/3$.