B1. A particle of mass m moves in a central potential

$$U(r) = -\frac{U_0}{e^{r/a} - 1}$$

where $U_0 \gg \dfrac{\hbar^2}{ma^2}$

a) (1 point) Give a semi-qualitative argument why for low lying energy levels $U(r)$ can be well approximated by a Coulomb-like potential $-U_0 a/r$.

b) (6 points) Use first order perturbation theory around the Coulomb potential to first order in r/a to determine the energies of the first three lowest lying energy levels.

b) (3 points) What are the degrees of degeneracy of each of these levels and the corresponding levels of the unperturbed (pure Coulomb) problem?

You might find the following formulas useful:

$$\langle n_r l m | \hat{r} | n_r l m \rangle = [3n^2 - l(l+1)]\frac{\hbar^2}{2m\alpha}$$

$$E_{nl} = -\frac{m\alpha^2}{2\hbar^2 n^2} \qquad n = n_r + l + 1$$

where $|n_r l m\rangle$ and E_{nl} are the eigenstates and the energies in a Coulomb potential $-\alpha/r$ with $\alpha > 0$.

B2. Consider a particle of mass m in the delta function potential

$$U(x) = -\alpha\delta(x)$$

where $\alpha > 0$

a) (2 points) Find the energy levels and eigenstates of the discrete part of the spectrum.

b) (3 points) Compute the product of uncertainties in the particle's coordinate and momentum $\Delta x \Delta p$ in these states. (Here $\Delta A = \sqrt{< \hat{A}^2 > - < \hat{A} >^2}$.) Does your answer "minimize" the uncertainty relationship, i.e. is it the minimum possible value of the product $\Delta x \Delta p$?

c) (5 points) Suppose at $t = 0$ the particle was in a state $\phi(x, t = 0) = \sqrt{\beta}e^{-\beta|x|}$ where β is a parameter with dimensions of an inverse length . Determine the probability $p_\infty(x)dx$ to find the particle on the interval $(x, x+dx)$ in the limit $t \to \infty$. (Hint: Consider the expansion in energy eigenstates.) Compute the integral $w = \int_{-\infty}^{+\infty} p_\infty(x)dx$. Is $w = 1$? Why or why not?

B3. Unpolarized light of wavelength $\lambda = 550nm$ and intensity I_0 with its direction of propagation along the z-axis passes through a linear polarizer in the $z = 0$ plane with its transmission axis along the x-axis. After passing through the polarizer, the light enters an optically active material whose indices of refraction for right circularly polarized light is n_R and for left circularly polarized light is n_L. The thickness of this material is $d = 1.2mm$ and it occupies the region between the $z = 0$ and $z = d$ planes. After passing through this material the light passes through a second linear polarizer in the $z = 2d$ plane with its transmission axis along the y axis.

a) (3 points) Find the intensity of light as a fraction of I_0 after passing through the first polarizer. Show your work.

b) (4 points) Find the polarization state of the light after it exits optically active material but before it enters the second polarizer. Assume $n_R - n_L = 7.1 \times 10^{-5}$.

c) (3 points) Find the intensity of the light as a fraction of I_0 after it passes through the second polarizer.

B4. Consider an electrically neutral plasma of N free electrons and N free protons at temperature $T = 0$. In this plasma the electrons can be non-relativistic or relativistic but the protons are always non-relativistic. The volume of the system is V

a) (2 points) Compute the Fermi energy of the electrons and protons. At approximately what particle number density $n = N/V$ do the electrons become semi-relativistic?

b) (1 point) Compare the electron pressure to the proton pressure in the non-relativistic limit. Would your result change if the electrons were relativistic ?

c) (3 points) Find the total electron energy for relativistic electrons. Leave your answer in terms of an integral over $x = p/m_e c$. Investigate the ultra-relativistic and non-relativistic limits $x_F \gg 1$ and $x_F \ll 1$.

d) (2 points) Find how the pressure P depends upon the number density $n = N/V$ in the non-relativistic ($x_F \ll 1$) and ultra-relativistic ($x_F \gg 1$) limits . In other words find the exponent α in the relation $P \sim \rho^\alpha$.

e) (2 points) Express the pressure P as a function of the total mass M and radius R using $V = \frac{4\pi}{3}R^3$ In other words find the exponents β and γ in the relation $P \sim M^\beta/R^\gamma$, again in the two limits $x_F \gg 1$ and $x_F \ll 1$.

B5. Consider an ideal non-relativistic gas of particles of spin s in a volume V at temperature T.

a) (2 points) Assuming classical statistics but including a Gibbs factorial, find the canonical partition function for N particles and the grand canonical partition function .

b) (2 points) Find the mean number of particles $< N >$ in terms of V, T, μ (chemical potential) in the grand canonical ensemble.

c) (3 points) Find the mean square fluctuation in the number of particles $< N^2 > - < N >^2$ in the grand canonical ensemble and discuss the limit of $\sqrt{< N^2 > - < N >^2}/ < N >$ as $V \to \infty$ keeping $< N > /V$ fixed.

d) (3 points) Find the entropy S and Helmholtz free energy F in this system and show that $S/ < N >$ and $F/ < N >$ are intensive quantities.

Possibly useful expressions:

$$N! \sim e^{-N} N^N$$

$$\int_{-\infty}^{+\infty} dx x^2 e^{-x^2} = \frac{\sqrt{\pi}}{2}$$