DAY TWO

In this exam you will have to answer four questions in Thermal Physics and four in Quantum Mechanics. Read the instructions carefully since in some cases, but not all, you will have a choice of questions. Each question should be answered in a separate bluebook with the question label (example: TB2) printed clearly on the front along with your code number.

Part T: Thermal Physics

Everyone should answer question TA1 below. (8 pts)

TA1

A container is divided into two compartments by a partition. The larger compartment contains 4 moles of N_2 gas at absolute temperature T and pressure P. The smaller compartment contains 1 mole of 0_2 gas at the same T and P. The partition is subsequently removed, and the gases mix and come into equilibrium. The gases obey the ideal gas law.

(a) [2 points] What will be the final T and P after equilibrium is reached? What are the partial pressures of 0_2 and N_2 after the partition is removed?

(b) [3 points] What will be the change in Gibbs free energy function G in terms of RT, where R is the gas constant?

(c) [3 points] What will be the change in entropy S in terms of R?

TA1 Solution

(a) For ideal gases the initial volumes are related by $V_{N_2} = 4V_{O_2}$, and the final temperature must equal the initial temperature. The nitrogen expands by a factor of 5/4, the oxygen by a factor of 5, so that the final partial pressures are 4/5 P and 1/5 P, respectively. The final total pressure is therefore identical to the initial pressure.

(b) Since the temperature does not change, for each constituent $dG = VdP = nRTdP/P$ and integrating $\Delta G = nRT(P_f/P_i)$. The total change in G is therefore the sum of these expressions for the two constiuents: $\Delta G = RT\ln(1/5) + 4RT\ln(4/5) = RT\ln(4^4/5^5) = -2.502RT$.

(c) Since $G = U - TS + PV$ and U, T, and PV for each constituent do not change, $\Delta S = -\Delta G/T = 2.502R$.

Answer one of the 2 questions TB1 or TB2. (12 pts)

TB1

The diameter of a helium atom is $2.6 \times 10^{-8} cm$. For helium gas at temperature 300°K and pressure $10^5 N/m^2$

(a) [4 points] What is the mean free path of the He atoms?

(b) [4 points] What is the average time between collisions?

If the container of helium is surrounded by a vacuum and the wall of this container has a tiny leak of area $10^{-10} cm^2$,

(c) [4 points] how many helium atoms leak out in one hour?

$k_B = 1.38 \times 10^{-23} J/K$

$M_{proton} = 1.66 \times 10^{-27} kg$

TB1 Solution

(a) The mean free path is $\ell = 1/(n\sigma) = 1/(n\pi d^2)$, where, assuming an ideal gas, $n = P/(kT)$ is the number of He atoms per volume and $\sigma = \pi d^2$ is the collision cross section. For T= 300°K and P = $10^5 N/m^2$ $n = 2.42 \times 10^{25}/m^3$ and $\ell = 1/(2.42 \times 10^{25}/m^3 \pi(2.6 \times 10^{-10}m)^2) = 1.94 \times 10^{-7}m$.

(b) The average time between collisions is this distance divided be the average speed of the atoms: $\tau = \ell/\bar{v}$ where from the Maxwell distribution the average speed is given by $\bar{v} = 0.92\sqrt{3kT/m} = 0.92\sqrt{3(1.38 \times 10^{-23}J/K)(300K)/(4(1.66 \times 10^{-27}kg))} = 1367m/s$, so $\tau = 1.94 \times 10^{-7}m/1.37 \times 10^3 m/s = 1.42 \times 10^{-10}s$.

(c) The outward flux of particles is $\phi = n\bar{v}/4 = 8.2 \times 10^27/(m^2 s)$ so the number of particles escaping is $n_{esc} = \phi A t = 8.2 \times 10^{27z}/(m^2 s) \times 10^{-14}m^2 \times 3600s = 3 \times 10^{17}$ atoms .

TB2

(a) [3 points] Show that the change in entropy ds of a system can be written as

$$dS = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V dT + \frac{1}{T}\left[\left(\frac{\partial U}{\partial V}\right)_T + P\right] dV.$$

(b) [3 points] Show that

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P.$$

(c) [3 points] Use this expression to evaluate $\left(\frac{\partial U}{\partial V}\right)_T$ for an ideal gas.

(d) [3 points] Evaluate $\left(\frac{\partial U}{\partial V}\right)_T$ for a van der Waals gas where $(P+a(\frac{N}{V})^2)(V-Nb) = NkT$, with a and b constant parameters .

TB2 Solution

(a) From $dU = TdS - PdV$ and $dU = (\partial U/\partial T)_V dT + (\partial U/\partial V)_T dV$ one has $S = (1/T)(\partial U/\partial T)_V dT + (1/T)((\partial U/\partial V)_T + P)dV$.

(b) Comparing this result to $dS = (\partial S/\partial V)_T dV + (\partial S/\partial T)_V dT$ one finds $(\partial S/\partial T)_V = (1/T)(\partial U/\partial T)_V$ and $(\partial S/\partial V)_T = (1/T)((\partial U/\partial V)_T + P)$. Then taking derivatives with respect to the "other" variable, and using the fact that these second derivatives are independent of order, one finds $(1/T(\frac{\partial^2 V}{\partial T \partial V} = -(1/T^2)(\frac{\partial U}{\partial V} + p) + (1/T)(\frac{\partial^2 U}{\partial V \partial T} + \frac{\partial P}{\partial T})$, giving $(\partial U/\partial V)_T = T(\partial P/\partial T)_V - P$

(c) For an ideal gas $PV = NkT$ and the two terms on the right hand side cancel, so $(\partial U/\partial V)_T = 0$.

(d) For a van der Waals gas $(\partial P/\partial T)_V = Nk/(V - Nb)$ and after a cancellation $(\partial U/\partial V)_T = a(N/V)^2$, a positive number proportional to the square of the number density.

Answer one of the 2 questions TC1 or TC2. (16 pts)

TC1

Consider a system whose energy levels are equally spaced at $0, \epsilon, 2\epsilon, 3\epsilon, \ldots$, with each level 3-fold degenerate.

(a) [8 points] Write down all possible macrostates for a system of 6 distinguishable non-interacting particles and total energy $E = 6\epsilon$.

(b) [4 points] Find the total number of macrostates.

(c) [4 points] Find the thermodynamic probability of each macrostate.

TC1 Solution

(a) The macro states can be labelled by a set of positive integers summing to 6. For example 4+1+1 means that one particle is in the n=4 level, two in the n=1 level, and the other three in the n=0 ground state. The different macro states are thus 6, 5+1, 4+2, 3+3, 3+2+1, 2+2+2, 4+1+1, 3+1+1+1, 2+2+1+1, 2+1+1+1+1, and 1+1+1+1+1+1+1.

(b) There are 11 states listed in (a).

(c) Since each level is g =3 -fold degenerate, and N = 6, the number of microstates for the macrostate with N_i particles in the ith energy level is

$$W = N! \prod_{i=1}^{6} g_i^{N_i} / \prod_{i=1}^{6} N_i! = 6!3^6 / \prod_{i=1}^{6} N_i!.$$

For the 4+1+1 macro-state, then $W = 6!3^6/(1!2!3!) = 60 \times 3^6$. It is easy to see that all W's will have the factor 3^6, and these will cancel in the end. The numbers $W/3^6$ for the macro-states listed, and in the same order as above, are 6, 30, 30, 15, 120, 20, 60, 60, 90, 30, 1. The sum of all these numbers is 462.

(d) The thermodynamic probability of a macrostate is $W/(3^6 462)$. for the 4+1+1 macrostate, for example, this is $60/462 = 0.1299$.

TC2

(a) [4 points] Derive an expression for the canonical partition function of N non-interacting, non-relativistic indistinguishable and spinless particles in a volume V.

(b) [4 points] Obtain an expression for the chemical potential for part (a). Use
$N! \approx N^N e^{-N}$.

(c) [4 points] Obtain an expression for the entropy of such a system.

(d) [4 points] How are these results changed if the particles each had spin s?

TC2 Solution

(a) The canonical partition function $Z_N = Z_1^N/N!$ where $Z_1 = (V/h^3)\int d^3p e^{-p^2/(2mk_BT)} = V/\lambda_T^3$, with $\lambda_T \equiv h/(2\pi m K_B T)^{1/2}$.

(b) $F_N = -k_BT\ln Z_N = -k_BT\ln((V/\lambda_T^3)^N/N!)$. Using the given approximation for N! this reduces to $F_N = -Nk_BT\ln(Ve/(\lambda_T N))$. The chemical potential is then $\mu = \partial F_N/\partial N = -k_BT\ln(Ve/(\lambda_T N)) + k_BT = -k_BT\ln(V/(\lambda_T N))$. Note that this is equivalent to $N = (V/h^3)\int d^3p e^{(\mu-E)/(k_BT)}$.

(c) $S = -(\partial F_N/\partial T)_{V,N} = Nk_B(\ln(Ve/(\lambda_T^3 N)) + 3/2) = Nk_B\ln(Ve^{5/2}/(\lambda_T^3 N))$.

(d) If the particles have spin s, each momentum state has a degeneracy of (2s+1). This means that $Z_1 \to Z_{1,s} = (2s+1)Z_1$ and thus that $Z_N \to Z_{N,s} = (2s+1)^N Z_N$. This has the consequences that $F_N \to F_{N,s} = F_N - k_BTN\ln(2s+1)$, $\mu \to \mu_s = \mu - k_BT\ln(2s+1)$, and $S \to S_s = S + k_BN\ln(2s+1)$. Not that the change in μ is exactly that required to cancel a multiplicative factor of $(2s+1)$ which must be added to the expression for N given at the end of the solution to part (b).

Answer one of the 2 questions TD1 or TD2.(20 pts)

TD1

One kilogram of water is heated by an electrical resistor from 20°C to 99°C at constant (atmospheric) pressure. Assume the heat capacity is constant in this temperature range. Estimate:
(a) [5 points] The change in internal energy of the water.
(b) [5 points] The entropy change of the water.
(c) [5 points] The factor by which the number of accessible quantum states of the water is increased.
(d) [5 points] The maximum mechanical work achievable by using this water

as heat reservoir to run an engine whose heat sink is at 20°C.

TD1 Solution

(a) The change in internal energy of the water is

ΔU = M cΔT = 1000 $\times$ 1 $\times$ 79 = 7.9
$\times$ 10^4 cal.

(b) The change in entropy is
ΔS = $\int \frac{Mc}{T}$ dT = M c ln $\frac{T_2}{T_1}$ = 239 cal/K.

(c) From Boltzmann's relation S = k ln Ω, we get
$\frac{\Omega_2}{\Omega_1} = \exp\left(\frac{\Delta S}{k}\right) = \exp(7 \times 10^{25})$.

(d) As work is done heat leaves the water and its temperature decreases. Taking this into account one finds that the maximum mechanical work available is

$$W_{max} = \int_{T_1}^{T_2} \left(1 - \frac{T_1}{T}\right) McdT = Mc(T_2 - T_1) - T_1 Mc\ ln\frac{T_2}{T_1} = 9 \times 10^3 cal.$$

TD2

In our three-dimensional universe, the following are well-known results from statistical mechanics and thermodynamics:

(a) [7 points] The energy density of black body radiation depends on the temperature as T^α, where $\alpha = 4$.

(b) [7 points] In the Debye model of a solid, the specific heat at low temperatures depends on the temperature as T^β, where $\beta = 3$.

(c) [6 points] The ratio of the specific heat at constant pressure to the specific heat at constant volume for a monatomic ideal gas is $\gamma = 5/3$.

Derive the analogous results (i.e., what are γ, α and β) in a universe with n spatial dimensions.

TD2 Solution

(a) The energy of black body radiation is

$$E = 2 \int\int \frac{d^n p d^n q}{(2\pi\hbar)^n} \frac{\hbar w}{e^{\hbar w/2\pi kT} - 1} = \frac{2V}{(2\pi\hbar)^n} \int d^n p \frac{\hbar w}{e^{\hbar w/2\pi kT} - 1}.$$

For the radiation we have p = $\hbar\omega$/c, so

$$\frac{E}{V} = 2 \left(\frac{k}{2\pi\hbar c}\right)^n k \int d^n x \frac{x}{e^x - 1} . T^{n+1},$$

where x = $\hbar$ ω/kT. Hence α = n + 1.

(b) The Debye Model regards solid as an isotropic continuous medium with partition function

$$Z(T, V) = \exp\left[-\hbar \sum_{i=1}^{nN} \omega_i / 2kT\right] \prod_{j=1}^{nN} [1 - \exp(-\hbar\omega_j/kT)]^{-1}.$$

The Helmholtz free energy is

$$F = -kT \ln\ Z = \frac{\hbar}{2} \sum_{i=1}^{nN} \omega_i + kT \sum_{i=1}^{nN} \ln[1 - \exp(-\hbar\omega_i/kT)].$$

When N is very large,

$$\sum_{i=1}^{nN} \rightarrow \frac{n^2 N}{\omega_D^n} \int_0^{\omega_D} \omega^{n-1} d\omega,$$

where ω_D is the Debye frequency. So we have

$$F = \frac{n^2 N}{2(n+1)}\hbar\omega_D + (kT)^{n+1}\frac{n^2 N}{(\hbar\omega_D)^n}\int_0^{x_D} x^{n-1}\ln[-\exp(-x)]dx,$$

where $xD = \hbar\omega_D/kT \to \infty$ at low temperatures. Hence

$$c_v = -T\left(\frac{\partial^2 F}{\partial T^2}\right)\infty T^n,$$

i.e., $\beta = n$.

(c) The theorem of equipartition of energy gives the constant volume specific heat of a molecule as $c_v = \frac{1}{2}k\imath$ where $\imath$ is the number of degrees of freedom of the molecule. For a monatomic molecule in a space of n dimensions, $\imath = n$. With $c_p = c_v + k$, we get

$$\gamma = \frac{c_p}{c_v} = \frac{(n+2)}{n}.$$

Part Q: Quantum Mechanics

Everyone should answer question QA1 below. (8 pts)

QA1

1 To investigate the properties of a sample of material, you decide to scatter neutrons that have a 1-nm wavelength.
For these neutrons find

(a) [2 points] the kinetic energy (in eV),

(b) [2 points] the wave vector (in m^{-1}),

(c) [2 points] the speed (give $\beta = v/c$), and

(d) [2 points] the momentum (in eV/c) .

It might be helpful to recall that $\hbar c = 197$ MeV·fm. (Approximate answers found without a calculator are acceptable.)

QA1 Solution

This problem is a straightforward application of the relations $k = 2\pi/\lambda$, $p = \hbar k$, $T = p^2/2m$, $v = p/m = \sqrt{2T/m}$, and $\nu = v/\lambda$, combined with the ability to put in appropriate factors of $\hbar$ and c to convert between systems of units. The only tricky aspect is that the relationship $E = h\nu$ might be thoughtlessly applied to derive the frequency; it applies only when $\beta = 1$.

(a) The neutron mass is about 1.67×10^{-27} kg $\rightarrow$ 940 MeV, so the neutron is non-relativistic, and its kinetic energy is $T = p^2/2m = 1.3\times10^{-22}$ kg·m^2/s^2 $\rightarrow$ 8.2×10^{-4} eV.

(b) The wave vector is given by $k = 2\pi/\lambda = 6.3\times10^9$ m^{-1} $\rightarrow$ 1.24×10^3 eV.

(c) The neutron speed is given by $v = p/m$, so $\beta = v/c = pc/mc^2 = 4.0 \times 10^2 m/s = 1.32 \times 10^{-6}$.

(d) The momentum is given by $p = \hbar k = 6.6\times10^{-25}$ kg·m/s $\rightarrow$ 1.24×10^3 eV/c.

Answer one of the 2 questions QB1 or QB2. (12 pts)

QB1

A particle of mass m is in the ground state of an infinite potential well of width a ($U = 0$ for $0 < x < a$ and $U = \infty$ otherwise). Suddenly the right wall of the well moves to a point $b > a$.

(a) [8 points] Determine the probability for a particle to remain in the ground state.

(b) [4 points] How does the answer change if the wall moved to its new position very slowly (adiabatically)?

You may find useful the indefinite integral

$$\int dx \sin(\alpha x) \sin(\beta x) = [\alpha \sin(\beta x) \cos(\alpha x) - \beta \sin(\alpha x) \cos(\beta x)]/(\beta^2 - \alpha^2).$$

QB1. Solution

(a) When the wall suddenly moves to the new position, the particle remains in the old ground state. The probability to find it in the new ground state is

$$P_{\text{sudden}} = |\langle a|b\rangle|^2$$

where $|l\rangle$ stands for the ground state of a particle in the infinite potential well of width l. Using

$$\langle x|l\rangle = \sqrt{\frac{2}{l}} \sin \frac{\pi x}{l}$$

we obtain

$$P_{\text{sudden}} = \frac{4ab^3}{\pi^2(a^2 - b^2)^2} \sin^2 \frac{\pi a}{b}.$$

(b) If the wall moves to its new position infinitely slowly, the particle stays in the ground state at all times, i.e.

$$P_{\text{adiabatic}} = 1$$

QB2

[12 points] A particle is in the ground state of an infinite square well, $V(x) = \infty$ for $|x| > a$, but $V(x) = 0$ for $|x| \leq a$. This simple potential is modified by the *small* perturbing potential $V_1(x) = V_0$ for $|x| > b$, but $V_1(x) = 0$ for $|x| \leq b$, with $b < a$. Estimate the change in energy ΔE of the ground state due to this perturbation. Verify that your formula for ΔE is correct in the limits $b \to 0$ and $b \to a$.

QB2. Solution

To do this problem, it helps to recall the indefinite integral

$$\int d\theta \cos^2\theta = \frac{\sin 2\theta}{4} + \frac{\theta}{2},$$

which is found using integration by parts and subsequently the replacement $\sin^2\theta = 1 - \cos^2\theta$ in the integral.

The ground state of the infinite square well potential is just the sinusoid curve $\psi_0(x) = c_0 \cos(\pi x/2a)$, with the normalization c_0. The coefficient c_0 is determined from

$$1 = \int_{-a}^{a} dx\, c_0^2 \cos^2(\pi x/2a).$$

Let $\theta = \pi x/2a$ and the limits are $-\pi/2$ to $\pi/2$. The integral becomes

$$1 = \frac{2a}{\pi} c_0^2 \left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_{-\pi/2}^{\pi/2},$$

giving the normalization coefficient $c_0 = \frac{1}{\sqrt{a}}$.

Perturbation theory tells us that the change in energy is the expectation value of the perturbation,

$$\Delta E = \int dx\, \psi_0(x)\, V_1(x)\, \psi_0(x).$$

Putting in the wave function and using the symmetry about $x = 0$ yields

$$\Delta E = \frac{2V_0}{a} \int_b^a dx\ \cos^2(\pi x/2a).$$

Thus,

$$\Delta E = \frac{2V_0}{a} \frac{2a}{\pi} \left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_{\pi b/2a}^{\pi/2}$$

$$\frac{\Delta E}{V_0} = \left[1 - \frac{\sin(b\pi/a)}{\pi} - \frac{b}{a}\right]$$

The intuitive limits of $\Delta E/V_0 \to 1$ as $b \to 0$, and $\Delta E \to 0$ as $b \to a$, are easily verified from the expression above.

Answer one of the 2 questions QC1 or QC2. (16 pts)

QC1

(a) [4 points] A one dimensional potential with a large narrow peak at position x_0 can be approximated by the delta function potential $V(x) = \alpha\delta(x - x_0)$. How do the wave function $\psi(x)$ and its first derivative $d\psi/dx$ change as one crosses the singular point x_0 ?

(b) [8 points] Suppose a particle of mass m moves in the one dimensional potential $V(x) = \alpha\delta(x) + \alpha\delta(x - a)$. Use the conditions you found in (a) to find an equation for the wave numbers k for which a plane wave incident from the left will not be reflected from this potential.

(c) [4 points] In the large α limit find the smallest value of k for which there is no reflection, and discuss the corresponding wave function.

QC1. Solution

(a) Using

$$\psi'(x_0 + \epsilon) - \psi'(x_0 - \epsilon) = \int_{x_0-\epsilon}^{x_0+\epsilon} \psi''(x)dx$$

and the Schrödinger equation it is easy to show that

$$\begin{array}{l} \psi(x_0 + 0) = \psi(x_0 - 0) \\ \psi'(x_0 + 0) - \psi'(x_0 - 0) = \frac{2m\alpha}{\hbar^2}\psi(x_0) \end{array},$$

the continuity of the wave function following becaus its derivative is everywhere finite.

(b) The wave function is e^{ikx}, $Ae^{ikx} + Be^{-ikx}$, and Ce^{ikx} in regions $x < 0$, $0 < x < a$, and $a < x$, respectively. This corresponds to having particles incident from the left and no reflected particles. Using the results in (a) at $x_0 = 0$ and $x_0 = a$, one obtains four equations for A, B, C, and k. Energies E for which there is no reflection are determined by the following transcendental equation:

$$\tan ka = -\frac{k\hbar^2}{\alpha m} \qquad E = \frac{\hbar^2 k^2}{2m}$$

(c) In the large α limit $\tan ka$ must vanish, and the smallest value of k for which this holds is $k = \pi/a$. The corresponding wave function is proportional to $\sin \pi x/a)$, which is just that for the infinite square well. This follows because as α increases the potential barriers become impenetrable.

QC2

Consider a particle of mass m in two dimensions in a potential $U(\rho) = -\alpha/\rho$, where $\alpha > 0$.

(a) [6 points] Assume the wave function can be written $\Psi(\rho, \phi) = \rho^\beta u(\rho) f(\phi)$. Find the possible forms for $f(\phi)$ and show that the exponent β can be chosen so that $u(\rho)$ satisfies a one-dimensional Schrödinger equation closely related to that for a hydrogen atom.

(b) [6 points] Find the discrete part of the energy spectrum.

(c) [4 points] What is the degree of degeneracy of each energy level?

QC2. Solution

(a) The Laplacian for cylindrical symmetry is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}.$$

The angular dependence is given by the usual $f(\phi) = exp(iM\phi)/\sqrt{2\pi}$, where M must be an integer so that f has the required 2π periodicity. The first ρ derivative term cancels provided one chooses $\beta = -1/2$, and then the Schrö dinger equation reduces to the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2 u}{d\rho^2} + \frac{\hbar^2}{2m}(M^2 - 1/4)u/\rho^2 - \frac{\alpha}{\rho}u = Eu.$$

This has exactly the form of the radial equation for the hydrogen atom except that there the centrifugal potential term is proportional to $\ell(\ell+1)$, instead of $M^2 - 1/4$, and of course there the wave function is proportional to u/r, not $u/\sqrt{\rho}$.

(b) Thus the results for the hydrogen atom can be used provided one replaces ℓ by $|M| - 1/2$. The wave functions can be labelled by M and a radial quantum number n_ρ, and the energy eigenvalues are given by

$$E_{n_\rho,M} = -E_H/(n_\rho + |M| + 1/2)^2,$$

where $E_H = m\alpha^2/(2\hbar^2)$ is the magnitude of the hydrogen ground state energy. These energy levels are thus $-4E_H/(2n+1)^2$, where $n = n_\rho + |M| = 0, 1, 2, \ldots$.

(c) Except for the ground state, these energy levels are all degenerate since different choices for n_ρ and M can produce the same n. It is easy to see that the degeneracy is $2n+1$ since for a given n one can have $n_\rho = 0$, $M = \pm n$, $n_\rho = 1$, $M = \pm(n-1)$, up to $n_\rho = n$, $M = 0$.

Answer one of the 2 questions QD1 or QD2. (20 pts)

QD1

(a) [12 points] Find the stationary states and the corresponding energies $\mathcal{E}$ for a charged spinless particle of mass m and charge e in constant uniform magnetic and electric fields that are perpendicular to each other. For definiteness assume the electric field $\mathbf{E}$ is in the +x direction with the magnetic field $\mathbf{H}$ in the +z direction. You can express your answer in terms of the eigenstates $\psi_n(\omega, x)$ of a one-dimensional harmonic oscillator of frequency ω. You should find ω in terms of e, m, and the field strengths. The complete wave function should depend upon the oscillator index n and two other parameters.

(b) [8 points] Find the average velocity of the particle in terms of n and the other parameters.

QD1. Solution

Choose the z- and the x-axis along the magnetic, H, and electric, E, fields, respectively. Write the vector potential as (Landau's gauge)

$$A_x = 0 \quad A_y = Hx \quad A_z = 0$$

The Hamiltonian reads

$$\hat{\mathcal{H}} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2m}\left(\hat{p}_y - \frac{eH}{c}x\right)^2 + \frac{\hat{p}_z^2}{2m} - eEx$$

where e is the charge. Since operators $\hat{p}_y$ and $\hat{p}_z$ commute with the Hamiltonian and with each other, the eigenstates can be chosen as

$$\Psi_{\mathcal{E}p_yp_z} = \frac{\exp(ip_y y + ip_z z)}{2\pi\hbar}\phi(x)$$

where $\mathcal{E}$ is the energy. The Schrödinger equation becomes

$$\phi''(x) + \frac{2m}{\hbar^2}\left[\mathcal{E} - \frac{p_z^2}{2m} + eEx - \frac{1}{2m}\left(p_y - \frac{eHx}{c}\right)^2\right]\phi(x) = 0$$

Completing the square in x, one obtains the Schrödinger equation for 1d harmonic oscillator that has a frequency $\omega_H = |e|H/mc$ and whose equilibrium is shifted to $x_0 = cp_y/eH + mc^2E/eH^2$.

Therefore,

$$\Psi_{np_yp_z} = \tfrac{\exp(ip_y y + ip_z z)}{2\pi\hbar}\psi_n(\omega_H, x - x_0)$$

$$\mathcal{E}_{np_yp_z} = \hbar\omega_H\left(n + \tfrac{1}{2}\right) - \tfrac{cEp_y}{H} - \tfrac{mc^2E^2}{2H^2} + \tfrac{p_z^2}{2m} \quad n = 0, 1, \ldots$$

The average velocity is

$$v_x = 0 \qquad v_z = \frac{\partial\mathcal{E}_{np_yp_z}}{\partial p_z} = \frac{p_z}{m} \qquad v_y = \frac{\partial\mathcal{E}_{np_yp_z}}{\partial p_y} = -\frac{E}{H}c.$$

One can also obtain the expression for $v_y = (\langle p_y - (eH/c)x\rangle)/m$ by noting that the average value of x is just x_0 and taking the expression above for x_0. Note the drift in the -y direction with the speed Ec/H. This is just the motion for which the classical forces on the particle cancel, as in a velocity selector.

QD2

Two perfectly spherical balls of radius r are positioned with one directly above the other, separated by a distance d, with $d >> r$. The lower ball is fixed in place. The top ball is released, and due to the gravitational force $F = mg$ drops and bounces elastically from the lower ball. Estimate the maximum number of times it can bounce from the lower ball before it misses. In particular:

(a) [8 points] Estimate the minimum horizontal offset between the centers of the balls at the time of the first bounce. (Consider only the position of the upper ball here; ignore quantum mechanical uncertainties to the position of the ball fixed in place, and to the sphericity of the balls.)

(b) [6 points] The collision imparts a small horizontal momentum to the upper ball. Estimate by what factor the horizontal offset and horizontal component of momentum grow with each successive collision.

(c) [6 points] Using $m_{ball} = 0.1$ kg, $r = 4$ cm, $d = 1$ m, $g \approx 10$ m/s^2, and $\hbar \approx 10^{-26}$ J·s, estimate the maximum number of bounces.

QD2. Solution

8a) The uncertainty principle limits the horizontal alignment of the balls to

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta p_x \, t/m)^2,$$

with the time it takes the ball to drop given by $t = \sqrt{2d/g}$. Using the uncertainty principle we replace $\Delta p_x \to \hbar/\Delta x$. We minimize Δs by setting the derivative equal to 0, to obtain $(\Delta x)^2 = \sqrt{\frac{2d}{g}} \frac{\hbar}{m}$. Thus, $(\Delta s)^2 = 2\frac{\hbar}{m}\sqrt{\frac{2d}{g}}$.

8b) Since $r << d$, we can use small angle approximations. The upper ball falls close to vertically, and its speed on impact is approximately given by $v_v = \sqrt{2gd}$. Assume - we confirm this below - that it is a good approximation on each bounce that the sideways offset of the upper ball is due almost entirely to the sideways velocity imparted by the previous collision. Then, collision n occurs at an offset of s_n. The angle from vertical between the centers of the balls is given by $\theta = s_n/2r$. After the collision, the ball bounces off at an angle of 2θ from vertical, with a sideways velocity $v_{xn}/v_v = s_n/r$ leading to $v_{xn} = \sqrt{2gd}(s_n/r)$. Collision $n+1$ takes place at an offset of $2tv_{xn}$, where $2t$ is the time between collisions, $2t = 2\sqrt{2d/g}$. Thus, $s_{n+1} = 2\sqrt{2d/g}\sqrt{2gd}(s_n/r) = (4d/r)s_n$. Similarly, the sideways velocity is increased by a factor of $4d/r$. Since $r << d$, $4d/r >> 1$, and the assumption holds.

8c) Using the results from parts a) and b), the n^{th} bounce is the last bounce if

$$\left(2\frac{\hbar}{m}\sqrt{\frac{2d}{g}}\right)^{1/2}(4d/r)^n \geq r.$$

Putting in numbers, $4d/r = 100$, and $\left(2\frac{\hbar}{m}\sqrt{\frac{2d}{g}}\right)^{1/2} \approx 3\times10^{-17}$, so we require $10^{2n} \geq 1.3 \times 10^{15}$. Thus, $n = 8$ is an upper limit on the number of bounces.