DAY ONE

In this exam you will have to answer four questions in Classical Mechanics and four in Electricity and Magnetism. Read the instructions carefully since in some cases, but not all, you will have a choice of questions. Each question should be answered in a separate bluebook with the question label (example: EB2) printed clearly on the front along with your code number.

Part M: Classical Mechanics

Everyone should answer question MA1 below.

MA1

[8 points] A small mass starts out at rest a great distance from the sun and falls freely in to it. How long will it take for the mass to travel from a distance from the center of the sun equal to the radius $r_\oplus$ of the earth's orbit to the surface of the sun? Assume the mass does not pass near the earth or any other planet, and let $R_\odot$ be the radius of the sun. Use information about the earth's orbit to express your answer as a numerical fraction of the year.

MA1 Solution

From conservation of energy the speed of the mass when it is at a distance r from the center of the sun is

$$v = \sqrt{2GM_\odot/r}$$

so that the time required to fall from $r_\oplus$ to $R_\odot$ is

$$\Delta t = \int_{R_\odot}^{r_\oplus} dr/v = \int_{R_\odot}^{r_\oplus} dr/\sqrt{2GM_\odot/r} = (2/3\sqrt{2GM_\odot})(r_\oplus^{3/2} - R_\odot^{3/2}).$$

Newton's Law or the Virial theorem applied to the earth's orbit gives

$$v_\oplus = 2\pi r_\oplus/T_\oplus = \sqrt{GM_\odot/r_\oplus}.$$

This can be used to replace the $r_\oplus^{3/2}$ term. Dropping the small $R_\odot^{3/2}$ term then gives

$$\Delta t = T_\oplus/(3\sqrt{2}\pi),$$

which is equivalent to about 27 days.

Answer one of the 2 questions MB1 or MB2.

MB1

(a) [6 points] Prove that for a particle of mass m in an arbitrary central force field the angular momentum $\mathbf{L}$ is conserved.

(b) [6 points] Express L in terms of m, the distance from the origin r and the angular velocity $\dot{\theta}$. Suppose the particle moves in the spiral orbit

$$r = c\theta^3.$$

How does θ vary with time?

MB1 Solution

(a) Since $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, Newton's 2nd law gives

$$d\mathbf{L}/\mathbf{dt} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F}.$$

Since $\mathbf{p}$ is in the same direction as $\mathbf{v}$ the first term vanishes, and therefore $\mathbf{L}$ is a constant if $\mathbf{F}$ is a central force in the same (or opposite) direction as $\mathbf{r}$ so that the second term also vanishes.

(b) Using the orbit equation

$$L = mr^2\dot{\theta} = mc^2\theta^6 d\theta/dt.$$

This can be integrated to give

$$Lt = mc^2\theta^7/7,$$

or

$$\theta = Kt^{1/7},$$

where $K = (7L/(mc^2))^{1/7}$.

MB2

A stationary space station can be approximated as a hollow spherical shell of mass 6 tonnes (6 000 kg), with inner and outer radii of 5m and 6m respectively. To change its orientation, a uniform flywheel (radius 10 cm, mass 10 kg) at the center of the ship is spun up quickly from rest to 1000 rpm.

(a) [6 points] How long will it take the station to rotate through 10°?

(b) [6 points] How much energy is needed to to spin up the flywheel?

MB2 Solution

(a) The formula for the solid sphere can be used to show that the moment of inertia of the ship is $I_s = \frac{2}{5}M_s(b^5 - a^5)/(b^3 - a^3)$, where $a = 6m$ and $b = 5m$, while the moment of inertial of the flywheel is $I_w = M_w c^2/2$, where $c = 0.10m$. Angular momentum conservation requires $I_s\omega_s = I_w\omega_w$. Inserting numbers, we obtain $\omega_s = 4.08 \times 10^{-4}$ rpm, and the ship will therefore take 68 minutes to make 1/36 of a full turn.

(b) The KE of the ship is negligible in comparison with that given the flywheel: $T_w = \frac{1}{2}I_w\omega_w^2$, because its angular speed is so low. Thus the energy needed is just 274 J.

Answer one of the 2 questions MC1 or MC2.

MC1

A particle is constrained to move without friction on a circular cone with half-angle α; the axis of the cone is vertical and the apex is down.

(a) [4 points] Write the Lagrangian, $\mathcal{L}$, in terms of spherical polar coordinates r and ϕ.

(b) [6 points] Find the two equations of motion. Show that the angular momentum, L_z, is conserved and use it to eliminate $\dot{\phi}$ from the r equation. What is the value of r_0 at which the particle can remain on a horizontal circular path?

(c) [6 points] Suppose the particle moves with small departures from this circular orbit: $r(t) = r_0 + \epsilon(t)$. Show, for small ϵ, that the particle executes stable harmonic oscillations about r_0, and find the frequency of the oscillation.

MC1 Solution

(a) The polar angle is fixed at $\theta = \alpha$, so

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2 \sin^2 \alpha \dot{\phi}^2) - mgr\cos\alpha.$$

(b) Since $\partial\mathcal{L}/\partial\phi = 0$, we have that $\partial\mathcal{L}/\partial\dot{\phi} = mr^2 \sin^2 \alpha\dot{\phi} = L_z$is constant. The r equation is

$$m\ddot{r} = mr\sin^2 \alpha\dot{\phi}^2 - mg\cos\alpha \qquad \text{or} \qquad \ddot{r} = \frac{L_z^2}{m^2 r^3 \sin^2 \alpha} - g\cos\alpha.$$

For a horizontal circular orbit $\ddot{r} = 0$, so $r = r_0 = [L_z^2/(m^2 g \sin^2 \alpha \cos\alpha)]^{1/3}$.

(c) Substituting $r = r_0 + \epsilon$ into the r equation of motion yields

$$\ddot{\epsilon} = \frac{L_z^2}{m^2 r_0^3 \sin^2 \alpha}\left(1 + \frac{\epsilon}{r_0}\right)^{-3} - g\cos\alpha$$

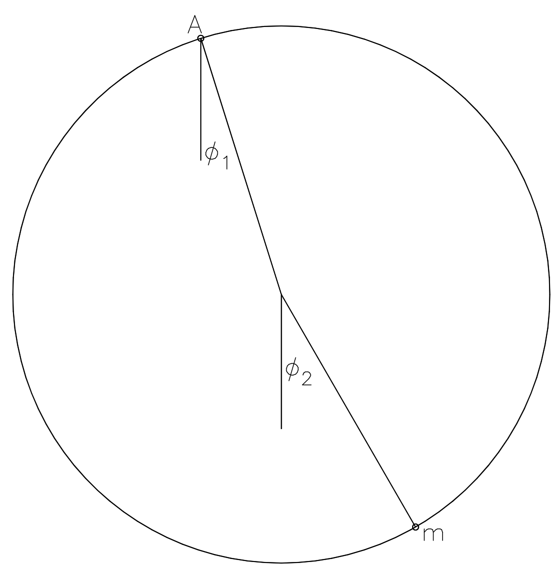
or for small ϵ/r_0

$$\ddot{\epsilon} \approx \frac{L_z^2}{m^2 r_0^3 \sin^2\alpha}\left(1 - \frac{3\epsilon}{r_0}\right) - g\cos\alpha = \frac{-3L_z^2}{m^2 r_0^4 \sin^2\alpha}\epsilon.$$

This harmonic oscillator equation has a real frequency, $\omega = \sqrt{3}L_z/(mr_0^2\sin\alpha)$, so the oscillations are stable.

MC2

[16 points] A bead of mass m is threaded on a frictionless circular wire hoop of radius R, which also has mass m. The hoop is suspended at the point A and is free to swing in its own vertical plane, as shown in the Figure. Using the angles ϕ_1 and ϕ_2 as generalized coordinates, solve for the normal frequencies of small oscillations. Find and describe the motion in the corresponding normal modes.



MC2 Solution

The moment of inertia of a uniform hoop about a point on the circumference is $I = 2mR^2$, so its KE is $T_1 = mR^2\dot{\phi}_1^2$. When the oscillations are small, the speed of the bead is $v_2 = R(\dot{\phi}_1 + \dot{\phi}_2)$, so its KE is $T_2 = mR^2(\dot{\phi}_1 + \dot{\phi}_2)^2/2$.

The potential energy is $U = mgR(1 - \cos\phi_1) + mgR[(1 - \cos\phi_1) + (1 - \cos\phi_2)] \approx \frac{1}{2}mgR(2\phi_1^2 + \phi_2^2)$ for small oscilations. The Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}mR^2(3\dot{\phi}_1^2 + 2\dot{\phi}_1\dot{\phi}_2 + \dot{\phi}_2^2) - \frac{1}{2}mgR(2\phi_1^2 + \phi_2^2)$.

The two Lagrange equations

$$3\ddot{\phi}_1 + \ddot{\phi}_2 = -\frac{2g}{R}\phi_1$$

and

$$\ddot{\phi}_1 + \ddot{\phi}_2 = -\frac{g}{R}\phi_2,$$

which can be written in matrix notation $\boldsymbol{M}\ddot{\boldsymbol{\phi}} = -\boldsymbol{K}\boldsymbol{\phi}$.

The normal frequencies are the determined by $\det(\boldsymbol{K} - \omega^2\boldsymbol{M}) = (2\omega^2 - \omega_0^2)(\omega^2 - 2\omega_0^2) = 0$, where $\omega_0 = \sqrt{g/R}$. Thus the normal frequencies are $\omega_1^2 = \frac{1}{2}\omega_0^2$ and $\omega_2^2 = 2\omega_0^2$. There are thus two normal modes:

Mode 1: Substituting $\omega = \omega_1$ into $(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{a} = 0$, we find $a_1 = a_2$. Here the two angles oscillate in phase with equal amplitude, and the bead does not slide relative to the hoop.

Mode 2: Substituting $\omega = \omega_2$ into $(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{a} = 0$, we find $a_2 = -2a_1$. In this mode the two angles oscillate exactly out of phase with the amplitude of ϕ_2 being twice that of ϕ_1, the bead moves the same distance but in the opposite direction as the center of the hoop, and the center of mass of the system does not move.

Answer one of the 2 questions MD1 or MD2.

MD1

[20 points] An object moving freely undr the action of an unusual central force $F(r)$ follows a precisely circular path that passes through the point $r = 0$. Determine the functional form of F. For definiteness take R to be the radius of the circular path, and assume the motion is in the $z = 0$ plane.

MD1 Solution

The equations of motion in the radial and azimuthal directions are

$$m(\ddot{r} - r\dot{\phi}^2) = F,$$

and

$$m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = 0,$$

where m is the mass of the spaceship. The 2nd equation is equivalent to the conservation of angular momentum $d(mr^2\dot{\phi})/dt = 0$, so we can set

$$\dot{\phi} = h/r^2,$$

where h is a constant equal to the angular momentum divided by m. If the spaceship moves in a circular orbit of radius R through the origin then

$$r = r(\phi) = 2R\cos\phi.$$

The time derivative gives

$$\dot{r} = -2R\sin\phi\dot{\phi} = -2Rh\sin\phi/r^2.$$

Taking the time derivative of this expression and using again the conservation of angular momentum and the equation for the orbit, one finds

$$\ddot{r} = -h^2(8R^2 - r^2)/r^5.$$

Using this together with $\dot{\phi} = h/r^2$ in the radial equation of motion gives

$$F = -8mh^2R^2/r^5,$$

which is the desired expression for the force. (Of course hR must be chosen to match the strength of the force to produce this orbit.)

This result can also be derived using the conservation of energy E and angular momentum $\ell = mh$ to derive the orbit equation

$$dr/d\phi = \pm(mr^2/\ell)\sqrt{(2/m)(E - V(r) - \ell^2/(2mr^2))},$$

where $V(r)$ is the potential energy. For the given circular orbit

$$dr/d\phi = -2R\sin\phi = \pm\sqrt{4R^2 - r^2}.$$

The orbit equation can then be solved to show that

$$V(r) = -2\ell^2 R^2/(mr^4) + \text{ terms independent of r},$$

and therfore

$$F = -dV/dr = -8\ell^2 R^2/(mr^5),$$

the same as above if r is replaced by ρ.

MD2

The rotation period of the Earth has been measured to lengthen at the rate of 1.6 milliseconds per century, which is due to a frictional torque caused by tides raised (mostly) by the Moon.

(a) [4 points] Explain in a few sentences the physical origin of the frictional torque. How is the Moon's orbit affected by the tides it exerts on the Earth?

(b) [8 points] Approximate the tidal bulges on Earth as two point masses on the Earth's surface, each having a fraction f of the Earth mass, and lying on an Earth diameter inclined at an angle θ to the line of centers between the Earth and the Moon. Working to only the lowest significant order, derive an approximate expression for the tidal torque on the Earth due to the attraction of the Moon. State clearly the approximations you make.

(c) [4 points] Equate this torque to that implied by the measured slow-down rate to estimate a value for the combination $f \sin\theta$.

(d) [4 points] Make a reasonable guess for the value of θ and use the fact that, if the Earth had no land masses, the tidal variation in the ocean depth would be about 0.5m peak to trough, to consider whether ocean tides alone are enough to supply the frictional drag.

The approximate masses of the Earth and Moon are 5.974×10^{24} kg and 7.35×10^{22} kg respectively, the Earth's radius is 6 378 km and the distance between the centers of the Earth and Moon is 3.84×10^{5} km.

MD2 Solution

(a) The tidal field of the Moon stretches the Earth along the line of centers, raising two almost equal tidal bulges on the side nearest the Moon and the side farthest from the Moon. Since the Earth's spin period is shorter than the Moon's orbit period, friction between the tidal bulges and the Earth drags the bulges to lie in a direction that leads the line of centers. The gravitational attraction of the Moon on the two bulges applies a torque that removes angular momentum from the Earth. Since angular momentum is conserved, the Moon gains and its mean distance from the Earth is slowly increasing.

(b) The gravitational attraction of the Moon on the two bulges is $fGMm/(D \pm R\cos\theta)^2$, where M is the Earth's mass, m is the Moon's mass, D is the distance between their centers, R is the radius of the Earth, and the bulge diameter makes an angle θ to the Earth-Moon direction. To lowest order, these forces differ from the mean force acting on the Earth by $\approx \mp 2fGMmR/D^3$, assuming $\cos\theta \approx 1$. The torque on the Earth arises from the components of these difference forces directed along the Earth's surface; i.e., the torque is $\approx 2R\sin\theta \cdot 2GfMmR/D^3$ to lowest order. This expression neglects the slight difference in angle between the directions to the Moon from the Earth's center and the positions of each bulge.

(c) The observed torque is $I\dot{\Omega}$, where I is the moment of inertia of the Earth. Assuming the Earth to be a uniform density sphere (not really true), its MoI is $2MR^2/5$. Thus we find

$$f\sin\theta \approx \frac{D^3\dot{\Omega}}{10Gm}.$$

If the Earth's spin period is $\tau = 24$ hrs and $\delta\tau = 1.6$ ms, the change in Ω is $\delta\Omega = -2\pi\delta\tau/\tau^2$ to an excellent approximation. Since $\delta t = 100$ yr, $\dot{\Omega} = \delta\Omega/\delta t \approx 4.26 \times 10^{-20}$ s^{-2}. Thus

$$f\sin\theta \approx 4.9 \times 10^{-8}.$$

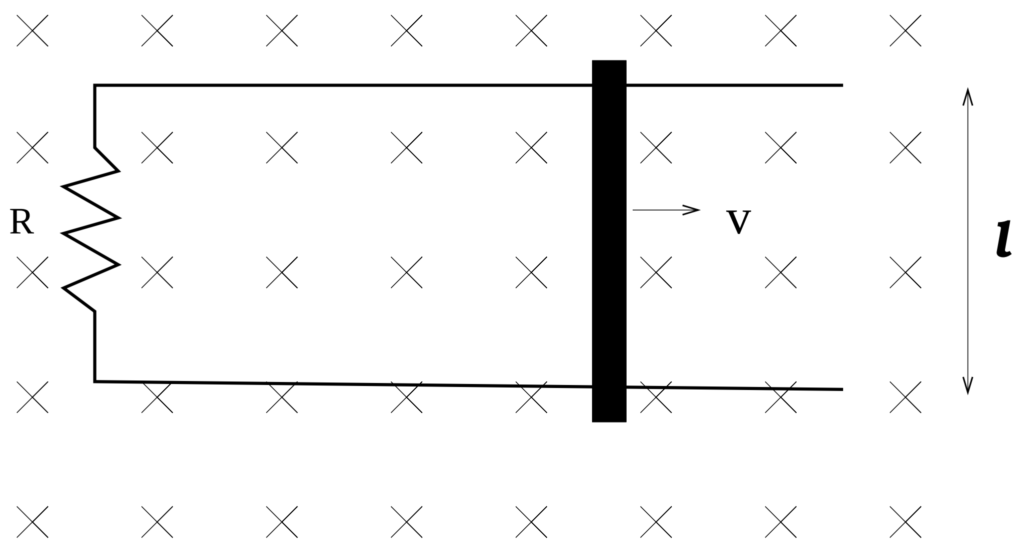
(d) Guessing $\theta \sim 3^\circ$, we find $f \sim 9.4 \times 10^{-7}$. We need an estimate of the mass of water in one ocean tidal bulge; for simplicity, assume that one bulge covers roughly one octant of the Earth's surface to a depth of $\delta R = 0.125$ m, which implies a mass $4\pi R^2 \delta R\rho/8$, with ρ being the density of water. This works out as 8×10^{15} kg, or $\sim 1.25 \times 10^{-9}$ Earth masses, implying that the ocean tides fail to supply the friction. (In fact most of the torque arises from distortions to the Earth itself.)

Part E: Electricity and Magnetism

Everyone should answer question EA1 below.

EA1

A metal bar slides without friction on two parallel conducting rails a distance ℓ apart as shown, with a resistor R connecting the two rails. A constant and uniform magnetic field B points into the page everywhere.



(a) [2 points] If the bar slides to the right at speed v, what current flows through the resistor? In what direction?

(b) [2 points] What is the magnetic force on the bar, and in what direction?

(c) [2 points] What is the power being dissipated in the resistor?

(d) [2 points]Discuss the conservation of energy in this system.

EA1 Solution

(a) If the area of the loop is a, then $da/dt = \ell v$. Then the electromotive force is

$$|\mathcal{E}| = \left|\frac{\partial \Phi}{\partial t}\right| = |B| \frac{da}{dt} = |B| \, \ell v. \tag{1}$$

The resulting current is $I = \mathcal{E}/R = B\ell v/R$.
Signs: If $\hat{n}$ is out of the page, then $\vec{B} = -B\hat{n}$ and positive $\mathcal{E}$ is counter-clockwise. Therefore, the current is counter-clockwise.

(b) The Lorentz force is

$$\vec{F} = I\vec{\ell} \times \vec{B} = -I\ell B\hat{x} = \left(\frac{B\ell v}{R}\right) \ell B = \frac{\ell^2 B^2 v}{R}. \tag{2}$$

The force is to the left.

(c) The power dissipated in the resistor is

$$P = I\mathcal{E} = (B\ell v)\left(\frac{B\ell v}{R}\right) = \frac{\ell^2 B^2 v^2}{R}. \tag{3}$$

(d) The mechanical power applied to the bar to keep it moving at constant velocity is

$$P = \vec{F} \cdot \vec{v} = \frac{\ell^2 B^2 v^2}{R}. \tag{4}$$

Thus, mechanical work is dissipated as heat in the resistor.

Answer one of the 2 questions EB1 or EB2.

EB1

A long, straight, uniform solid circular cylinder of conductivity σ has a radius R. Its axis coincides with the z axis of a coordinate system and it extends from $z = -\infty$ to $z = \infty$. Suppose that a uniform current I in the positive z direction passes through the cylinder. Define clearly what system of units you are using.

(a) [6 points] Calculate the magnetic field $\vec{H}$, in terms of the cylindrical coordinates r, ϕ, and z (r measures the distance from the the z axis, and ϕ is the azimuthal angle measured from the x axis).

(b) [6 points] Find the Poynting vector $\vec{S}$ at the surface of the cylinder. Relate it to the Ohmic heating per unit length in the cylinder.

EB1 Solution The solution for this problem uses the mksA or SI version of Maxwell's equations.

(a) Applying Ampere's Law, $\int \vec{H} \cdot d\vec{l} = I_{\rm enc}$, taking the contour to be a circle of radius r, whose center is on the z axis and is perpendicular to it, where $I_{\rm enc}$ is the current (conduction plus displacement) passing through the circle of radius r. Here $I_{\rm enc} = I$, for $r \geq R$ and $I_{\rm enc} = Ir^2/R^2$ otherwise. $\vec{H}$ will only have components in the $+\hat{\phi}$ direction given by $H_\phi = I_{\rm enc}/2\pi r$. Thus $H = Ir/(2\pi R^2)$ for $r \leq R$ and $H = I/(2\pi r)$ for $r \geq R$.

(b) The uniform electric field E in the cylinder is determined by the local version of Ohm's Law $J = I/(\pi R^2) = \sigma E$, in the +z direction. The Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ is in the inward normal direction, with magnitude $S = EH = I^2/(2\pi^2 R^3 \sigma)$, so that the power per length flowing into the surface of the cyliner is $2\pi RS = I^2/(\pi R^2 \sigma)$. Since the cylinder's resistance per length is $1/(\pi R^2 \sigma)$ this is equal to the Ohmic heating per unit length.

EB2

A dielectric sphere of radius R has electric polarization (dipole moment per volume) $\mathbf{P} = Ar^2\hat{\mathbf{r}}$, where A is a constant and $\mathbf{r} = r\hat{\mathbf{r}}$. There is no free charge.

(a) [4 points] Find the bound charge per volume $\rho_b(r)$ in the sphere and the bound charge per area σ_b on its surface.

(b) [4 points] Verify that the net bound charge in and on the sphere is zero.

(c) [4 points] Find the electric field inside ($r < R$) and outside ($r > R$) of the sphere.

EB2 Solution

Since there is no free charge anywhere $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = 0$ everywhere. Therefore

$$\rho_b = \epsilon_0 \nabla \cdot \mathbf{E} = -\nabla \cdot \mathbf{P}.$$

Evaluating the divergence for the given $\mathbf{P}$ gives

$$\rho_b = -4rA$$

in the interior of the sphere. Outside of the sphere $\mathbf{P}$ and $\mathbf{E}$ both vanish, so $\mathbf{E}$ is discontinuous at the surface. Gauss Law then requires that the surface charge is

$$\sigma_b = 4AR^2.$$

The net surface charge is just $4\pi R^2$ times this, while the net charge in the interior of the sphere is

$$4\pi \int_0^R \rho_b r^2 dr = -4\pi AR^4,$$

exactly the opposite of the net surface charge.

Answer one of the 2 questions EC1 or EC2.

EC1

A parallel plate capacitor has circular plates of radius R separated by a distance d. For times $t < 0$ the capacitor is uncharged. At $t = 0$ it is connected to a current source which delivers a linearly increasing current $I = \alpha t$ to the capacitor, where α is a constant. To fix the directions, assume the plates are horizontal and the positive charge flows into the lower plate and out of the upper plate.

(a) [8 points] Find the electric field $\mathbf{E}$ between the plates as a function of time.

(b) [8 points] Find the magnetic field $\mathbf{B}$ between the plates as a function of time.

Specify the directions of the fields as well as their magnitudes. You may ignore fringe effects and work in the quasi-static approximation, ignoring any contribution to $\mathbf{E}$ due to changes in $\mathbf{B}$.

EC1 Solution

(a) The charge density on the plates at time t is $\alpha t^2/(2\pi R^2)$ so that the electric field is in the upward vertical direction with magnitude $E = \alpha t^2/(2\pi\epsilon_0 R^2)$.

(b) Perhaps the simplest way to find the magnetic field is to note that the displacement current will be equal to the real current I spread out uniformly over the volume between the plates, so that the current density is just $I/(2\pi R^2)$. The problem is then equivalent to finding the magnetic field inside a fat wire of radius R. Using Ampere's Law one finds $B = \mu_0 r I/(2\pi R^2) = \mu_0 r\alpha t/(2\pi R^2)$. The magnetic field lines form horizontal concentric circles centered at the center of the plate and flow counterclockwise as seen from above.

EC2 An electromagnetic standing wave has its electric field

$$\mathbf{E}(z,t) = E_0 \hat{\mathbf{i}} \sin(kz)\cos(\omega t),$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors in the x, y, and z directions.

(a) [4 points] Show that this expression for $\mathbf{E}$ can be written as the superposition of two plane waves travelling in opposite directions.

(b) [6 points] What is the associated $\mathbf{B}$ field for this standing wave?

(c) [6 points] Find the Poynting vector $\mathbf{S}$ for the standing wave and use it to discuss the flow of electromagnetic energy.

You may find useful the identity $\sin A + \sin B = 2\sin((A+B)/2)\cos((A-B)/2)$.

EC2 Solution

NE5Soln (a) Using the identity

$$\mathbf{E}(z,t) = (E_0/2)\hat{\mathbf{i}} \sin(kz - \omega t) + (E_0/2)\hat{\mathbf{i}} \sin(kz - \omega t),$$

just the superposition of two plane wave, one travelling in the z direction, the other in the -z direction, and both polarized in the x direction.

(b) Each of the plane electromagnetic waves must have a magnetic field of magnitude $E_0/(2c)$ and in the $\pm$y direction with the sign chosen to give the Poynting vector in the correct direction.

$$\mathbf{B}(z,t) = (E_0/(2c)\hat{\mathbf{j}} \sin(kz - \omega t) - (E_0/(2c)\hat{\mathbf{j}} \sin(kz + \omega t).$$

Using the identity again this can be written

$$\mathbf{B}(z,t) = -(E_0/c)\hat{\mathbf{j}} \cos(kz)\sin(\omega t).$$

It is easy to check that the E and B fields satisfy the Maxwell equations $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \partial \mathbf{E}/\partial t$, and in fact one can also find B from E by requiring that these equations be satisfied.

(c) Using $2\sin\theta\cos\theta = \sin 2\theta$ one can show that

$$\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0 = -\hat{\mathbf{k}}\sin(2kx)\sin(2\omega t)/(4c\mu_0).$$

The time average of $\mathbf{S}$ is therefore zero, with the energy moving back and forth without a net flow.

Answer one of the 2 questions ED1 or ED2.

ED1

A spherical shell of radius R has a surface charge density, whose magnitude depends only on the polar angle θ from the z axis

$$\sigma(\theta) = \sigma_0 \cos\theta.$$

Be sure to specify your choice of units.

(a) [8 points] Is the electric field $\mathbf{E}$ continuous or discontinuous at $r = R$? If discontinuous discuss quantitatively the nature of the discontinuity. Repeat for the electrostatic potential Φ.

(b) [6 points] Find the potential Φ both inside ($r < R$) and outside ($r > R$) the the shell.

(c) [6 points] Find the electric field $\mathbf{E}$ both inside ($r < R$) and outside ($r > R$) the the shell.

ED1Soln

(a) From Gauss' Law for a small short cylindical volume enclosing a small portion of the shell one finds that the radial component of the electric field must be discontinuous with

$$E_r(R+, \theta) - E_r(R-, \theta) = \sigma(\theta)/\epsilon_0 = \sigma_0 \cos\theta/\epsilon_0.$$

Since $\mathbf{E} = -\nabla\Phi$ is finite everywhere, Φ must be continuous everywhere, even at $r = R$.

(b) The potential must satify Laplace's equation everywhere except on the shell. For any such region including the origin and azimuthally symmetric solution must have the form

$$\Phi = \sum_\ell a_\ell r^\ell P_\ell(\cos\theta),$$

while for a region including infinity it must have the form

$$\Phi = \sum_\ell a_\ell P_\ell(\cos\theta)/r^{\ell+1}.$$

Because the charge distribution is proportional t0 $\cos\theta = P_1(\cos\theta)$, only the $\ell = 1$ term contributes, and requiring continuity at $r = R$ means that

$$\Phi = \Phi_0(R^2/r^2)\cos\theta$$

for $r > R$ and

$$\Phi = \Phi_0(r/R)\cos\theta$$

for $r < R$. The constant Φ_0 can be determined from the discontinuity in the radial electric field:

$$E_r(R+,\theta) - E_r(R-,\theta) = 3\Phi_0\cos\theta/R = \sigma_0\cos\theta/\epsilon_0$$

, requiring $\Phi_0 = \sigma_0 R/(3\epsilon_0)$., so that the potential is now completely determined.

Another way to find the potential is to use

$$\Phi(\vec{r}) = k\int d^3r'\,\frac{\rho(\vec{r}\,')}{|\vec{r}\,' - \vec{r}|},$$

and then the expansion

$$\frac{1}{|\vec{r}\,' - \vec{r}|} = \sum_\ell r_<^\ell P_\ell(\cos\theta)/r_>^{\ell+1},$$

, where θ is the angle between the two vectors. Since only the $\ell = 1$ term contributes, it is easy to show find the same expression for Φ as above.

(c) We now have only to use $E_r = -\partial\Phi/\partial r$ and $E_\theta = -(1/r)\partial\Phi/\partial\theta$. to find the electric field everywhere. Outside the sphere we have $E_r = 2\sigma_0 R^3\sin\theta/(3\epsilon_0 r^3$ and $E_\theta = \sigma_0 R^3\cos\theta/(3\epsilon_0 r^3$, while inside the sphere $E_r = -\sigma_0\cos\theta/(3\epsilon_0)$ and $E_\theta = \sigma_0\sin\theta/(3\epsilon_0)$. Note that the field inside the sphere is uniform, with magnitude $\sigma_0/(e\epsilon_0)$ and in the -z direction.

ED2

This problem is meant to deduce the momentum and angular momentum properties of radiation and does not necessarily represent any real physical system of interest. Consider a charge Q in a viscous medium where the viscous force is proportional to the velocity: $\vec{F}_{visc} = -\beta\vec{v}$. Suppose a *circularly polarized* electromagnetic wave passes through the medium. The equation of motion of the charge is

$$m\frac{d\vec{v}}{dt} = \vec{F}_{visc} + \vec{F}_{Lorentz}. \tag{5}$$

Assume that the terms on the right dominate the inertial term on the left, so that approximately

$$0 = \vec{F}_{visc} + \vec{F}_{Lorentz}. \tag{6}$$

Let the frequency of the wave be ω and the strength of the electric field be E.

(a) [5 points] Show that to lowest order (neglecting the magnetic force) the charge moves on a circle in a plane normal to the direction of propagation of the wave with speed QE/β and with radius $QE/(\beta\omega)$.

(b) [5 points] Show that the power transmitted to the fluid by the wave is Q^2E^2/β.

(c) [5 points] By considering the small magnetic force acting on the particle, show that the momentum per unit time (force) given to the fluid by the wave is in the direction of propagation and has the magnitude $Q^2E^2/(\beta c)$.

(d) [5 points] Show that the angular momentum per unit time (torque) given to the fluid by the wave is in the direction of propagation and has magnitude $\pm Q^2E^2/(\beta\omega)$, where $(\pm)$ is for $\begin{pmatrix}\text{left}\\ \text{right}\end{pmatrix}$ circular polarization.

ED2 Solution

1. Substituting $\vec{F}_{Lorentz} = Q\vec{E}$ and $\vec{F}_{visc} = \beta\vec{v}$ into the force equation gives $v = QE/\beta$. The direction of the velocity rotates uniformly in a plane normal the propagation direction with period $2\pi/\omega$. Thus, the radius is found from

$$2\pi r = \oint v dt \tag{7}$$

to be $r = QE/(\beta\omega)$.

2. The power dissipated is $P = -\vec{v} \cdot \vec{F}_{visc} = \beta v^2 = Q^2E^2/\beta$. Since the orbit of the charge is constant in time, this is the power transmitted to the fluid.

3. The magnetic force is in the direction of propagation and has magnitude $F_{mag} = QBv/c = QEv/c = Q^2E^2/(\beta c)$. Here we have used $\left|\vec{E}\right| = \left|\vec{B}\right|$ for a free wave.

4. Using the center of the charge's motion as an origin, the magnitude of the torque is $\tau = \left|\vec{F}_{Lorentz} \times \vec{r}\right| = Q^2E^2/(\beta\omega)$. For a left-hand circularly polarized wave, the E-vector and, thus, the charge rotates counter-clockwise as viewed facing the wave. This imparts a torque *along* the direction of propagation. The opposite holds for right-hand polarization. Thus, $\tau = \pm Q^2E^2/(\beta\omega)$.

The above results can be used to deduce the momentum and angular momentum of circularly polarized photons with energy $E_\gamma = \hbar\omega$.