Solution for Problem A9

a) (5 points) The vector potential may be found from the integral (see, for instance, Marion and Heald, Classical Electromagnetic Radiation, Chapter 8):

\[
\vec{A}(\vec{x}, t) = \frac{1}{c} \int_V dV' \int dt' \frac{\vec{J}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \delta [t' + (|\vec{x} - \vec{x}'| - ct)/c].
\]  

(1)

The current density may be written with a complex time dependence (taking the real part at the end of the calculation):

\[
\vec{J}(\vec{x}, t) = I_o \hat{z} \cos \left(\frac{2\pi x}{\lambda}\right) \delta(x) \delta(y) e^{i\omega t}
\]

(2)

Substituting (2) into (1) and integrating over \(x'\) and \(y'\), we obtain

\[
\vec{A}(\vec{x}, t) \approx \frac{I_o \hat{z}}{cr} \int_{-\lambda/4}^{\lambda/4} dz' \cos \left(\frac{2\pi z'}{\lambda}\right) e^{i\omega (t - (1/c)\sqrt{x^2 + y^2 + (z-z')^2})},
\]

(3)

where we have used the assumption that we are in the radiation zone, so that

\[|\vec{x} - \vec{x}'| \approx r.\]

Expanding the square root in (3) to order \(z'/r\), we find

\[
\vec{A}(\vec{x}, t) \approx \frac{I_o \hat{z}}{cr} e^{i(\omega t - kr)} \int_{-\lambda/4}^{\lambda/4} dz' \cos \left(\frac{2\pi z'}{\lambda}\right) e^{i(2\pi z'/\lambda r)}. \]

(4)

Letting \(z = r \cos \theta\) and performing the integral in (4) (write cos as sum of exponentials), we get

\[
\vec{A}(\vec{x}, t) \approx \frac{I_o \hat{z}}{cr \pi} \frac{\lambda}{2} \left[ \frac{\sin((\pi/2)(1 + \cos \theta))}{1 + \cos \theta} + \frac{\sin((\pi/2)(1 - \cos \theta))}{1 - \cos \theta} \right] e^{i(\omega t - kr)}. \]

(5)

b) (3 points) The electric field \(\vec{E}\) in the radiation zone may be found directly from

\[
\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -i \frac{\omega}{c} \vec{A}
\]

using (5). The magnetic induction \(\vec{B}\) in the radiation zone is given by

\[
\vec{B} = -\frac{1}{c} \vec{n} \times \frac{\partial \vec{A}}{\partial t} = -i k (\vec{n} \times \vec{A}).
\]