Say ``Halo!'' to New Indices & New Walls

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OUTLINE

Introduction & Review

Part I: How BPS statespaces change with moduli in N=2 sugra

Part II: Line Operators in N=2 Field Theories

Framed BPS states & Protected Spin Characters

New derivation of the ``motivic KSWCF''

Exact results for line operator vevs in $T_{g,n}[A_1]$ theories

Old Question I

Strings2007: Introduced the semi-primitive WCF.

Final Riddle: Why did the BPS state cross the wall?

We need to understand not just the index but how the space of BPS states change as moduli are varied.

→ *New walls*

Old Question II

Strings2008: Moduli space $\mathcal{M}$ of an $\mathcal{N}=2$ theory on $\mathbb{R}^3 \times S^1$

Darboux/Twistor coordinates $\mathcal{X}_\gamma$ construct a HK metric

Final Promise:

These have an interpretation in terms of line operators

Part II: Make good on that promise

N=2: Basic Definitions

Moduli of vacua: $\mathcal{B}_{vm} \times \mathcal{M}_{hm}$

Local system: $\Gamma \to \mathcal{B}_{vm}$

$$\mathcal{H} = \oplus_{\gamma\in\Gamma}\mathcal{H}_{\gamma} \supset \oplus_{\gamma} \; h \otimes \mathcal{H}_{\gamma}^{\mathrm{BPS}}$$

$$Z : \Gamma \to \mathbb{C} \qquad\qquad E \geq |Z(\gamma; u)|$$

$$\langle \cdot , \cdot \rangle : \Gamma \to \mathbb{Z}$$

Old Indices & Old Walls

$\mathcal{H}_{\gamma}^{\mathrm{BPS}}$ Finite dimensional representation of $SU(2)_{\mathrm{space}}$

As such: Completely determined by their spin character:

$$s(\gamma, y; m) := \mathrm{Tr}_{\mathcal{H}_{\gamma}^{\mathrm{BPS}}} y^{2J_3}$$

This is not an index: It depends on $m \in \mathcal{B}_{vm} \times \mathcal{M}_{hm}$

Better: $\Omega(\gamma; u) := s\big|_{y=-1}$

Piecewise constant but can change across:

$$MS(\gamma_1, \gamma_2) := \{u | Z_1 \parallel Z_2\}$$

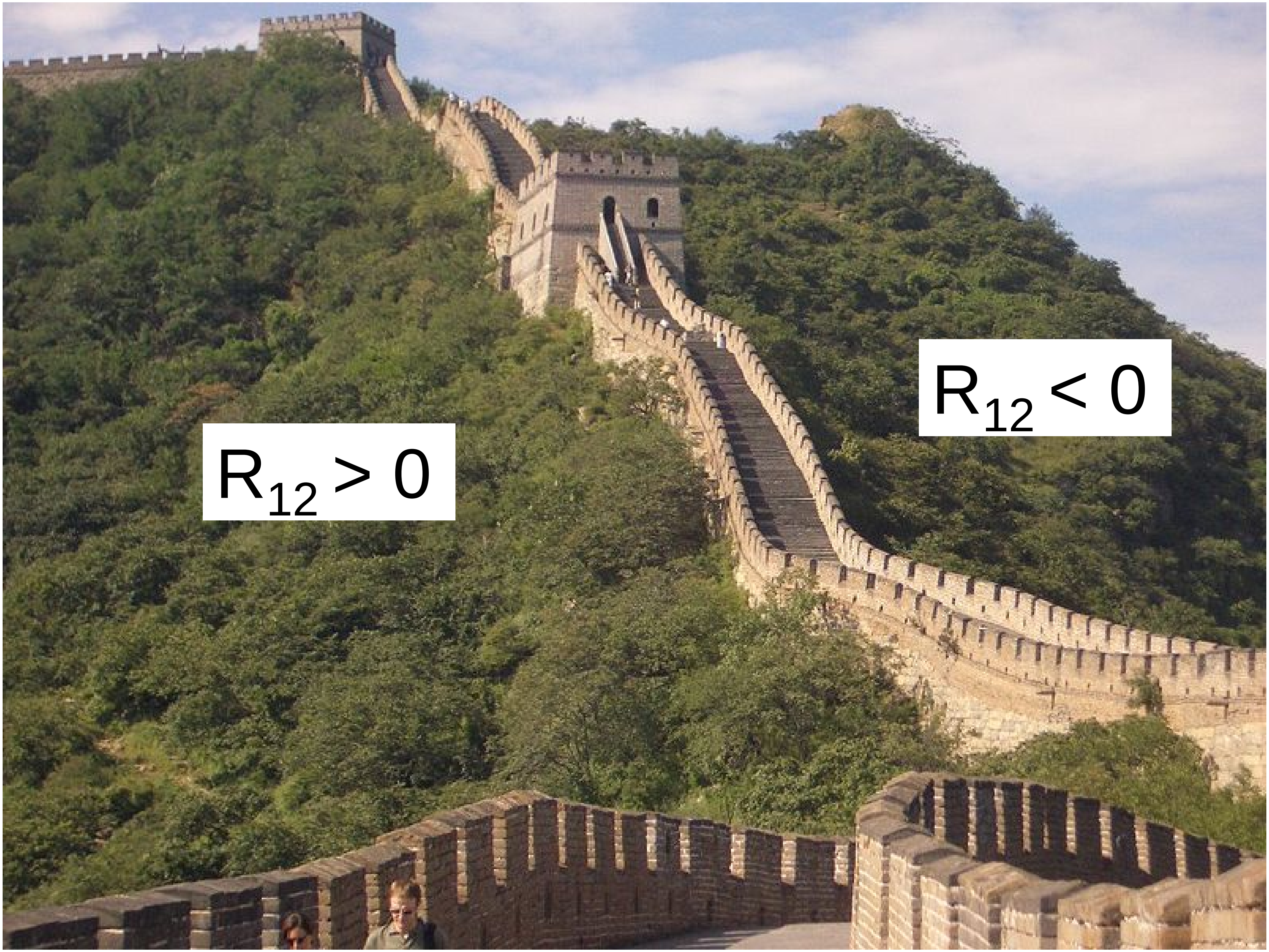
Old Boundstates

Boundstate radius (Denef)

$$R_{12} = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{2\mathrm{Im}(Z_1 Z_2^*)}$$

So the moduli space of vacua is divided into two regions:

$\langle \gamma_1, \gamma_2 \rangle \mathrm{Im}(Z_1 Z_2^*) > 0$ **OR** $\langle \gamma_1, \gamma_2 \rangle \mathrm{Im}(Z_1 Z_2^*) < 0$



$R_{12} < 0$
$R_{12} > 0$

Primitive Wall-Crossing

$$R_{12} = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{2\mathrm{Im}(Z_1 Z_2^*)}$$

Crossing the wall: $\mathrm{Im}(Z_1 Z_2^*) \to 0$

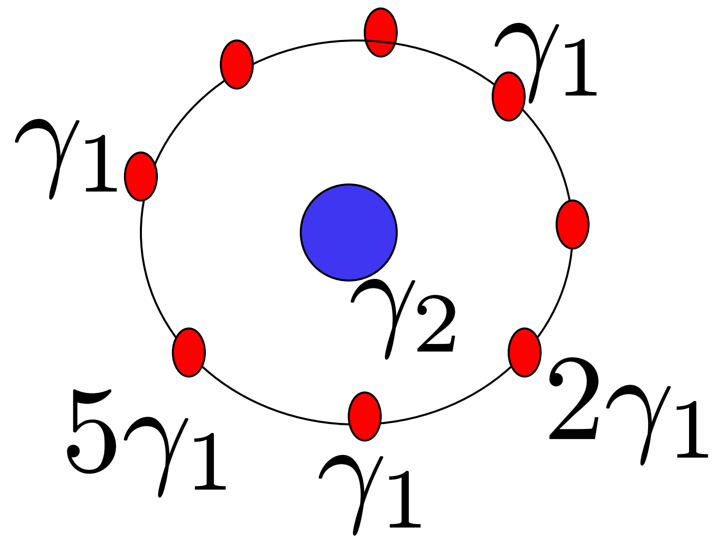
$$\boxed{\Delta \mathcal{H} = (J_{12}) \otimes \mathcal{H}(\gamma_1) \otimes \mathcal{H}(\gamma_2)}$$

$$2J_{12} + 1 = |\langle \gamma_1, \gamma_2 \rangle|$$

Old Halos

HYPERS → **Fermions**

VECTORS → **Bosons**



$$\mathcal{H}_{halo}^{BPS} := \mathcal{H}_{\gamma_2}^{BPS} \otimes_{\ell \geq 1} \mathcal{F}\,\underbrace{\left[(J_{\gamma_2, \ell\gamma_1}) \otimes \mathcal{H}(\ell\gamma_1) \right]}_{\text{Creation Operators}}$$

$$R_{\mathrm{halo}} \to \infty \quad \text{across} \quad MS(\gamma_1, \gamma_2)$$

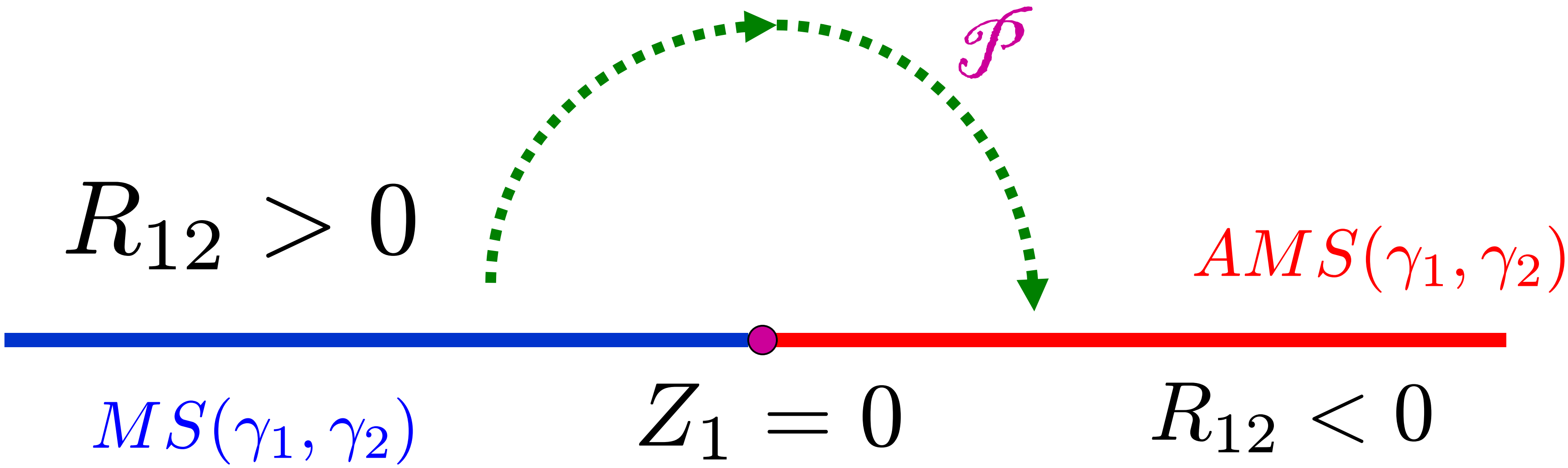
Part I

Systematic description of how BPS state-spaces change

arXiv:10??.???

E. Andriyash, F. Denef, D. Jafferis,

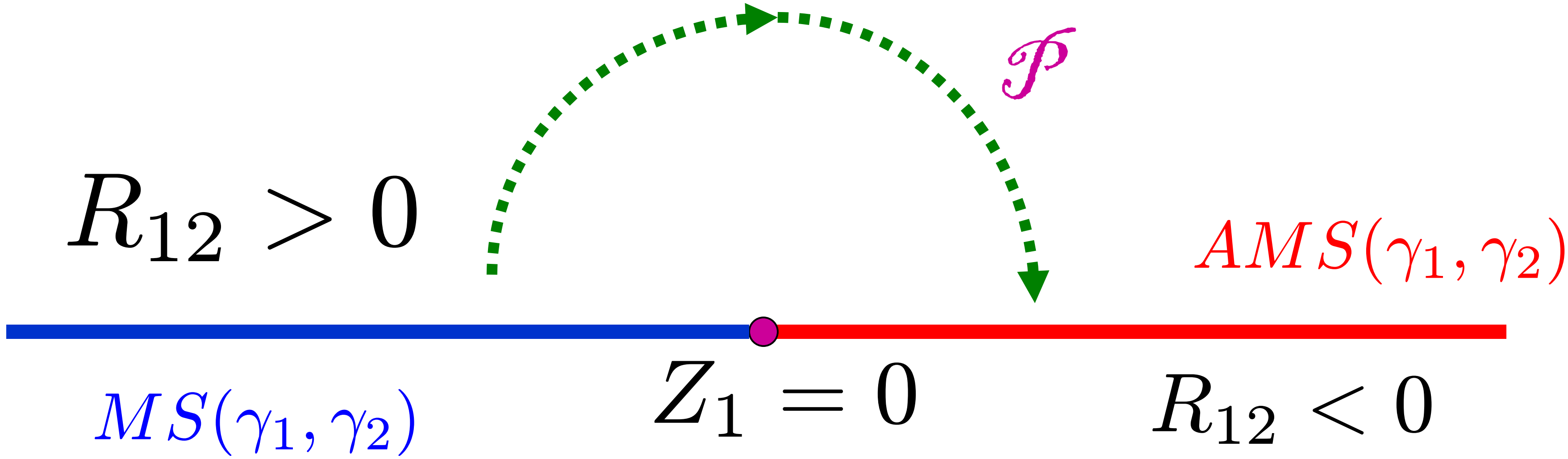
An Old Puzzle



$\mathscr{P}$ crosses no other walls of MS

Boundstate (γ_1, γ_2) exists near $MS(\gamma_1, \gamma_2)$

Boundstate (γ_1, γ_2) cannot exist near $AMS(\gamma_1, \gamma_2)$!

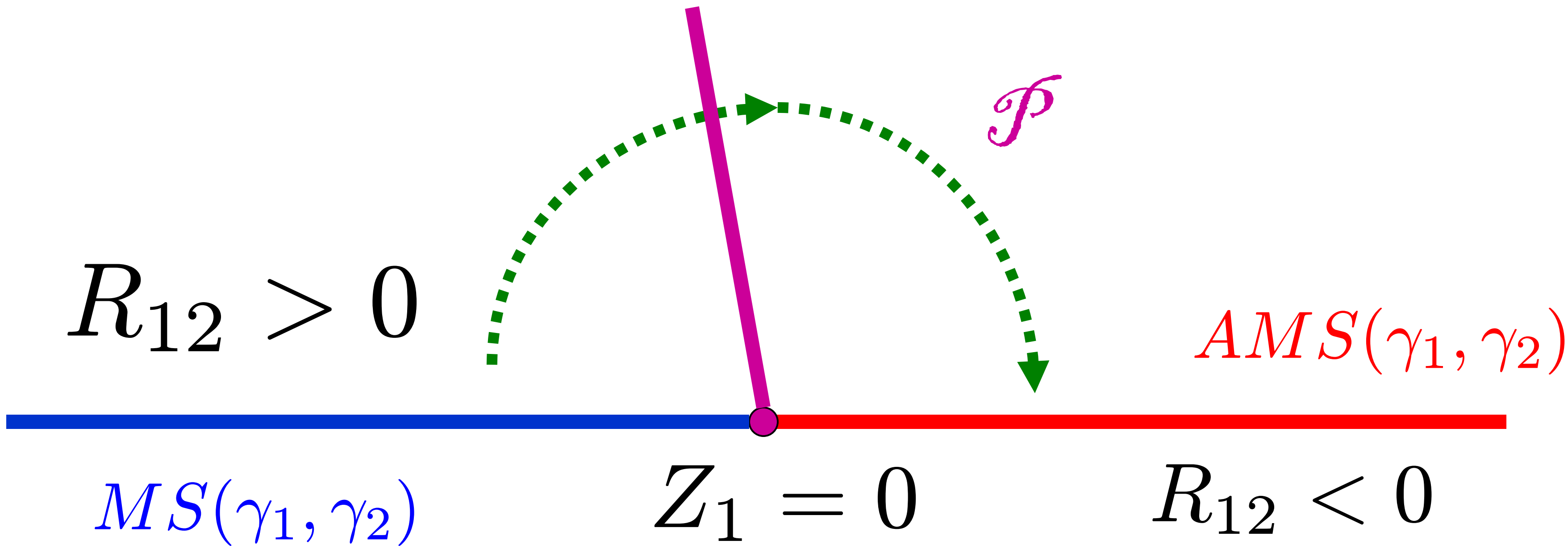


What happened?

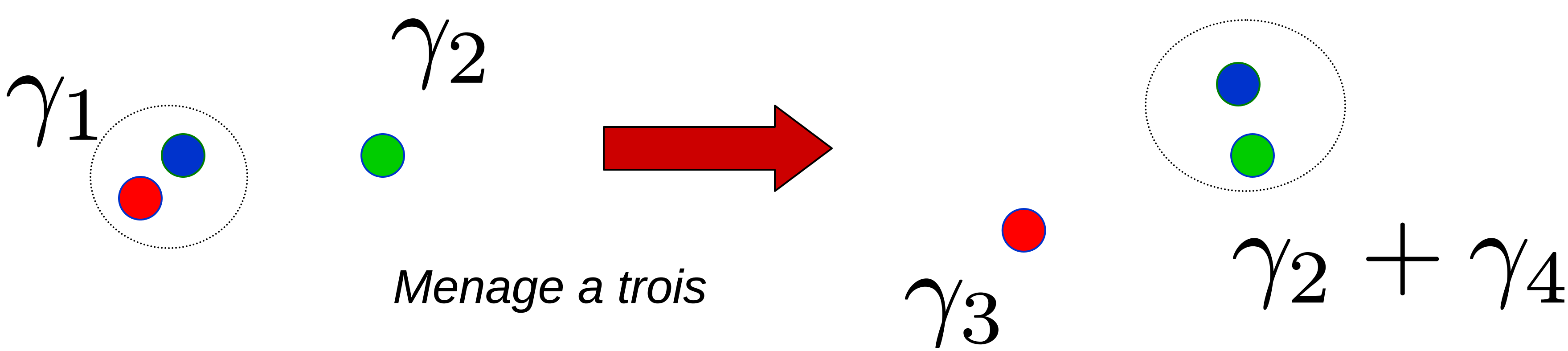
Just got married… VM + HM pair up

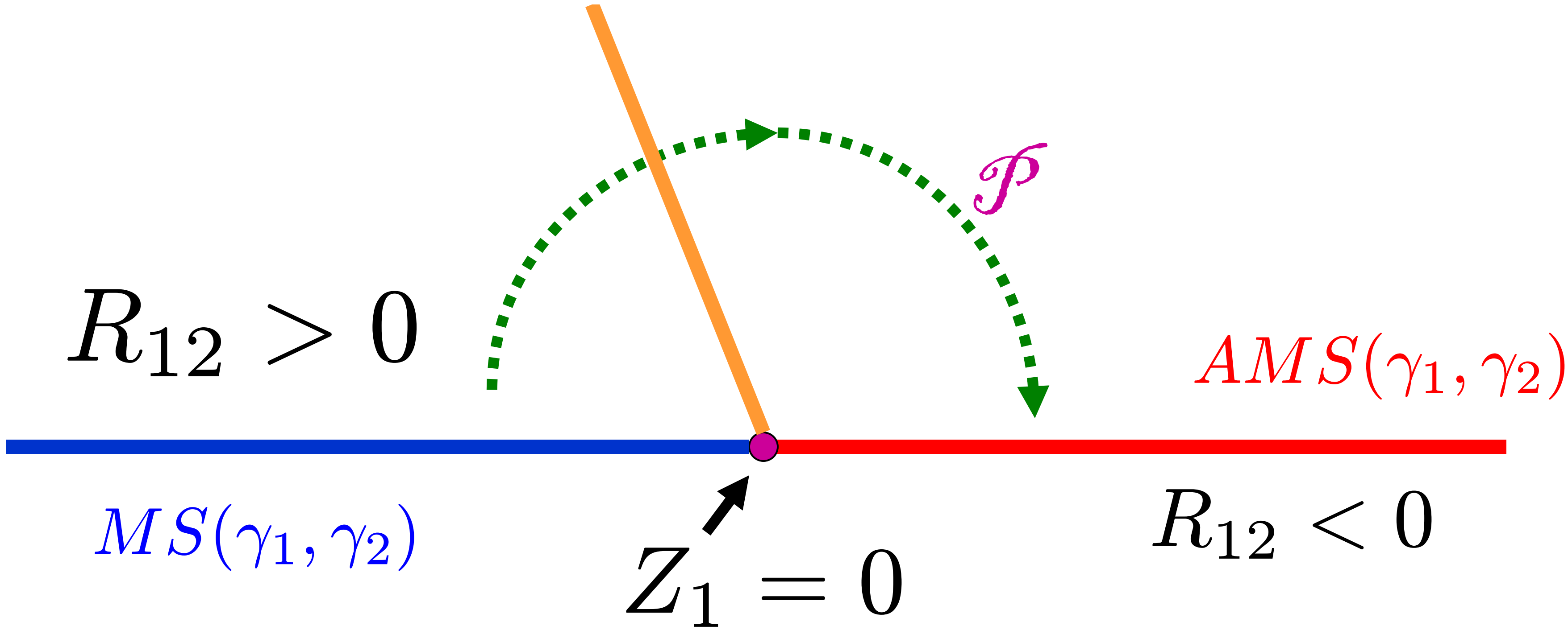
$\Delta\Omega_{vm} + \Delta\Omega_{hm} = 0$ (Note: $\Delta s \neq 0$)

Not the whole story: (γ_1, γ_2) boundstates contribute to $\Omega \neq 0$



$\mathcal{H}(\gamma_1) = 0$ near $Z_1 = 0$ $\Longrightarrow$ Recombination wall

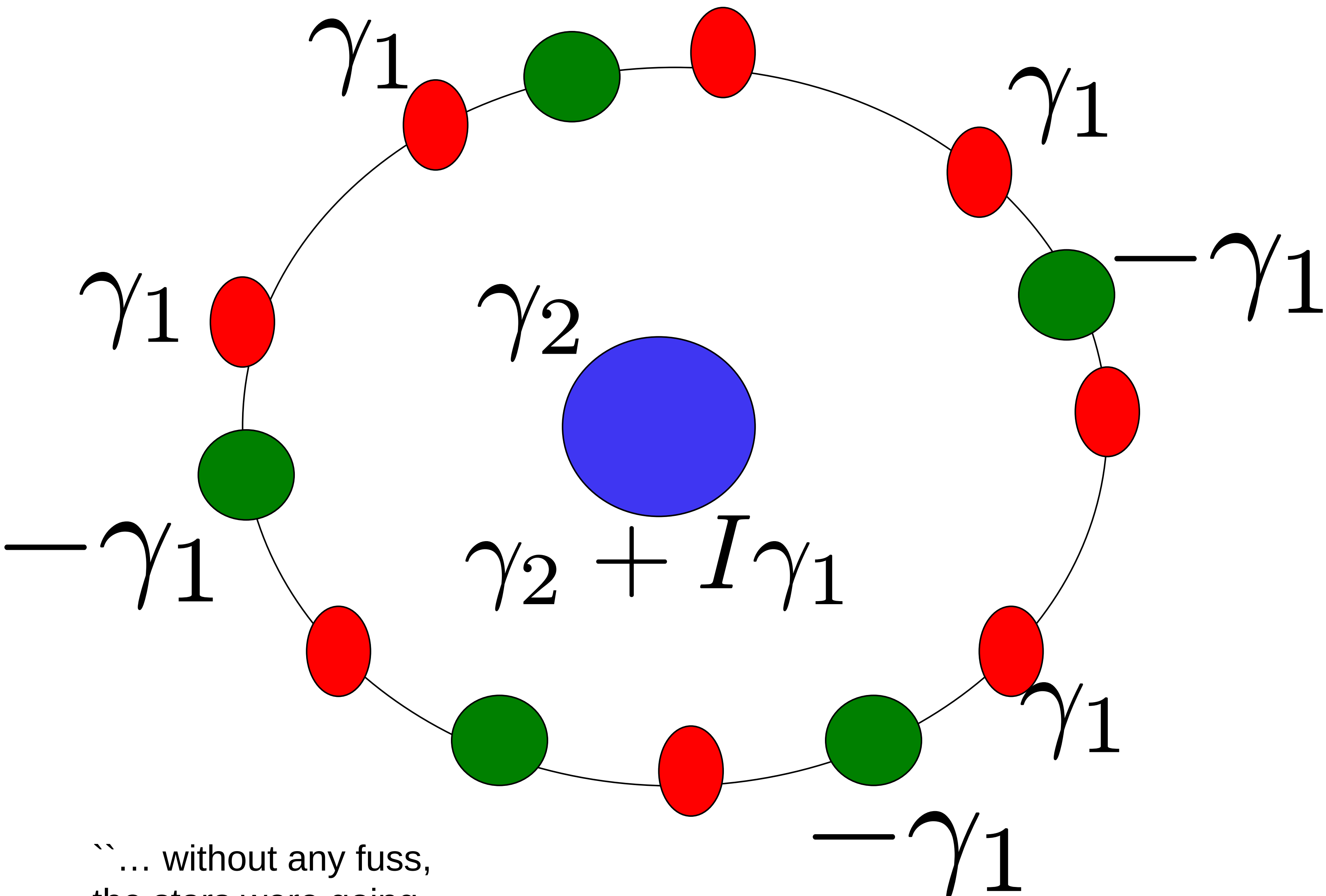




$\mathcal{H}(\gamma_1) \neq 0$ $\Longrightarrow$ New wall:

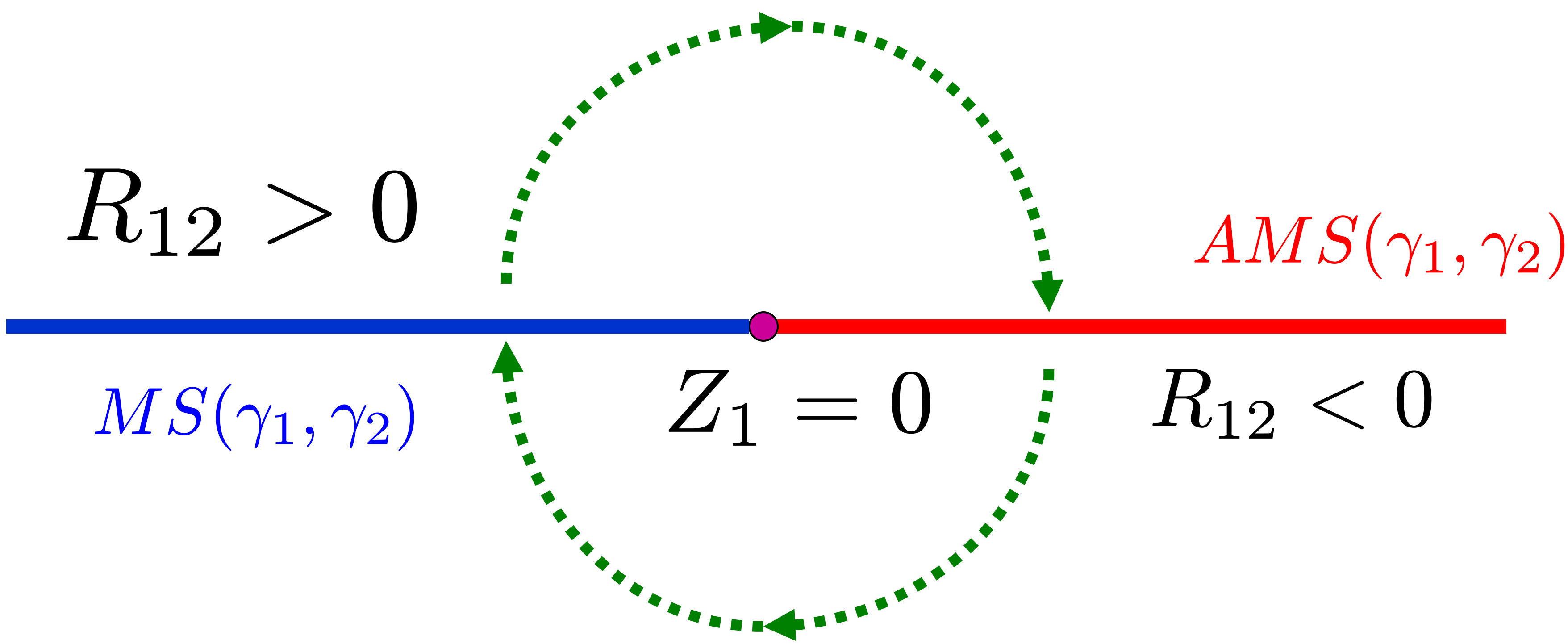
$\{u|(\gamma_1 + \gamma_2)$ attractor flow crashes on $Z_1 = 0\}$

$$A_{\rm probe} = \int (m ds + \langle \gamma, \mathcal{A} \rangle) \propto |Z(\gamma_1, u(\vec{x}))|$$



``… without any fuss,
the stars were going
out.'' – Arthur C.Clarke

Monodromy



$$\gamma_2 \to \gamma_2 + I\gamma_1$$

$$I = \langle \gamma_2, \gamma_1 \rangle \sum_{\ell=1}^{\infty} \ell^2 \Omega(\ell \gamma_1)$$

Further Predictions

The halo picture makes some further predictions about the spectrum of light states near a singular point of moduli space.

$$\sum_{\ell=1}^{\infty} \ell^2 \Omega(\ell\gamma_1) > 0$$

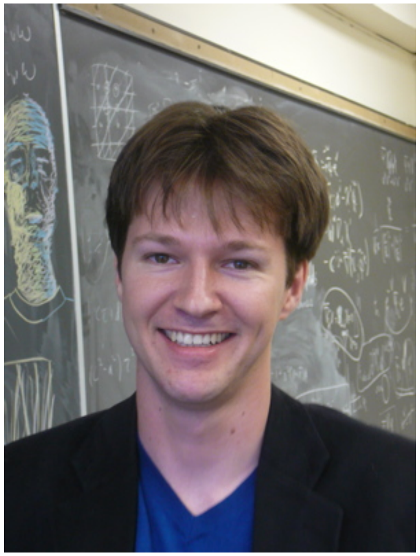
$$\prod_{\ell>0}(1-(-1)^{\ell|\langle\gamma_2,\gamma_1\rangle|} q^{\ell})^{\ell|\langle\gamma_2,\gamma_1\rangle|\Omega(\ell\gamma_1)}$$

These appear to contradict some of the literature on geometric engineering and extremal transitions.

We're trying to sort it out.

Part II: N=2 Field Theories

Davide Gaiotto & Andy Neitzke



arXiv:0807.4723 – Hyperkahler metrics and Darboux coordinates

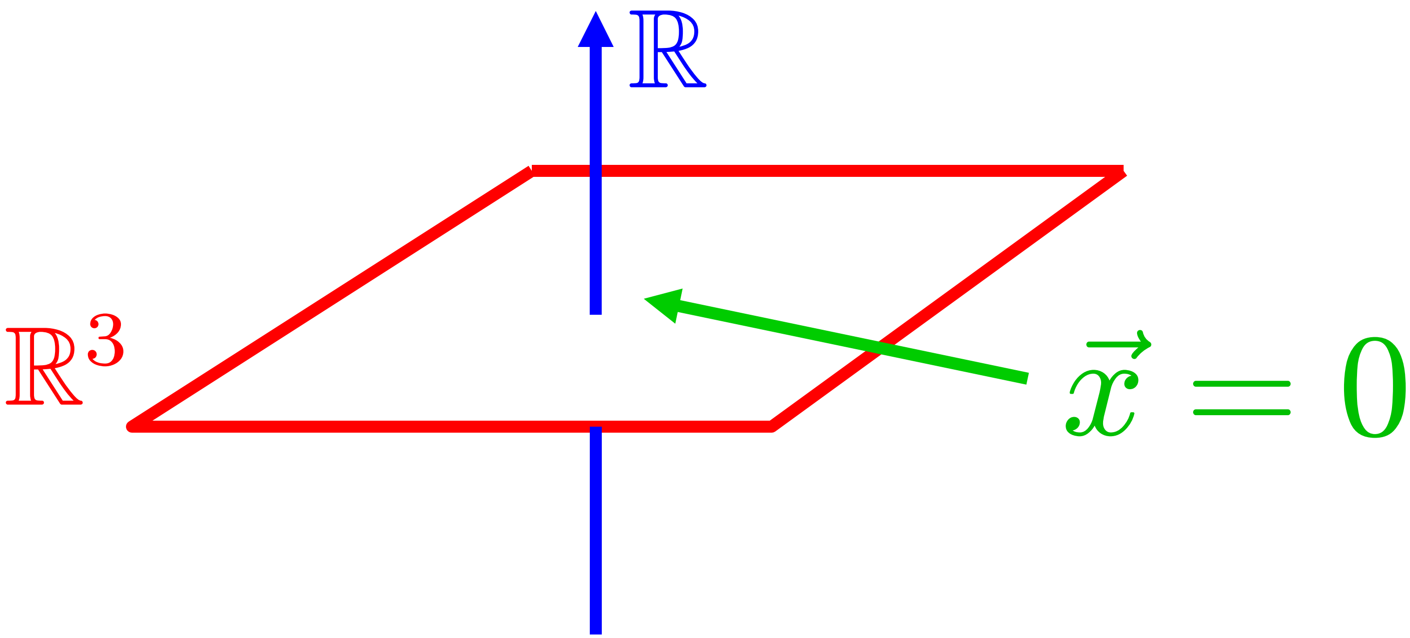
arXiv:0907.3987 BPS Spectrum and Darboux Coordinates for $T_{g,n}[A_1]$

Line Operators & Laminations, arXiv:10??.???

Line Operators

Now focus on d=4 N=2 field theory defined by some su(2,2|2) superconformal fixed point S.

Line operator = boundary condition for S on $AdS_2 \times S^2$ [Kapustin]



Unbroken Susy

Restrict attention to line op's preserving $osp(4^*|2)_\zeta$

Fixed points of an involution of su(2,2|2)

$$\vec{x} \to -\vec{x} \quad \& \quad U(1)_R \text{ rotation by } \zeta$$

$$osp(4^*|2)_\zeta^{\text{even}} = sl(2,\mathbb{R}) \oplus so(3) \oplus su(2)$$

Spatial rotation

R-symmetry

$$\mathcal{R}_\alpha^A \sim Q_\alpha^A + \zeta \sigma^0_{\alpha\dot\beta} \bar{Q}^{\dot\beta A}$$

$\Rightarrow$ Line operator L of type ζ $\quad L_\zeta(\cdots)$

New BPS Bound

Choose a line operator L preserving $osp(4^*|2)_\zeta$

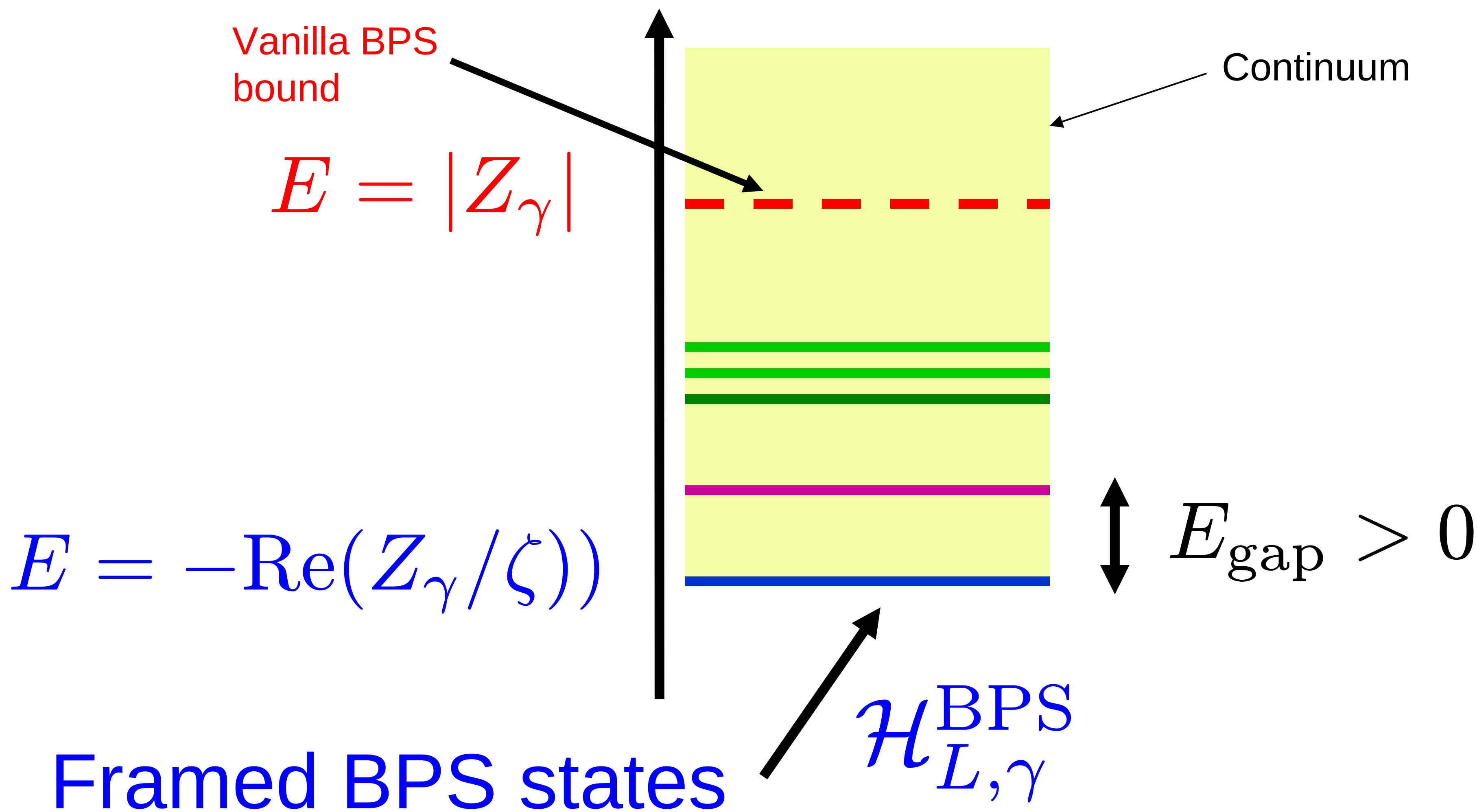
$\mathcal{H}_L$ Hilbert space in presence of L

$$\mathcal{H}_L = \oplus_\gamma \mathcal{H}_{L,\gamma}$$

$$\{\mathcal{R}_\alpha^A, \mathcal{R}_\beta^B\} = 4\epsilon_{\alpha\beta}\epsilon^{AB}(E + \mathrm{Re}(Z_\gamma/\zeta))$$

$$\Longrightarrow \quad \boxed{E \geq -\mathrm{Re}(Z_\gamma/\zeta)}$$

Framed BPS States



Protected Spin Character

Framed PSC for framed BPS states:

$$\boxed{\overline{\underline{\Omega}}(L,\gamma;y) := \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}_{L,\gamma}} y^{2J_3}(-y)^{2I_3}}$$

(Thanks to Juan Maldacena for an important suggestion.)

This is an index!

Vanilla PSC for vanilla BPS states:

$$\Omega(\gamma;y) := \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}_{\gamma}} y^{2J_3}(-y)^{2I_3}$$

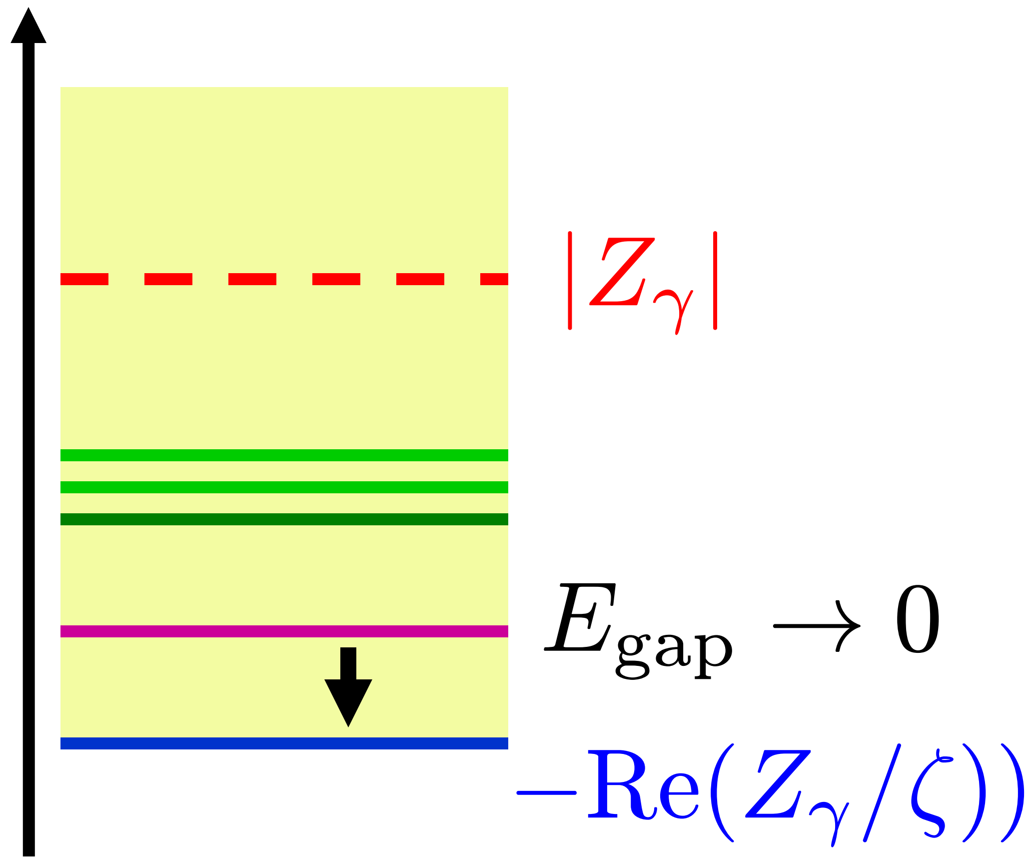
Also an index.

Closing the Gap

Gap can close
when

$$Z_{\gamma_h}/\zeta \in \mathbb{R}_-$$

for <u>some</u> BPS
charge



DEF: BPS ray:

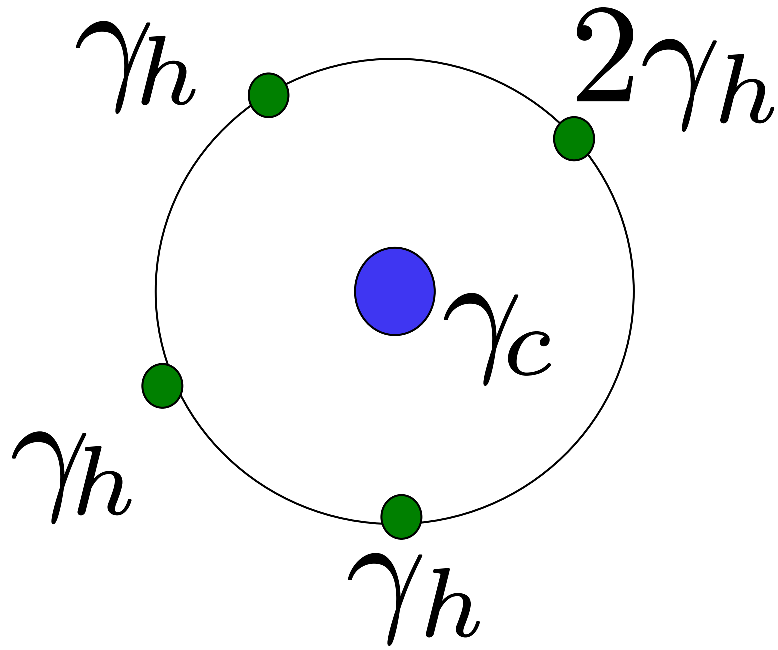
$$\ell_{\gamma_h,u} = \{\zeta | Z_{\gamma_h}/\zeta \in \mathbb{R}_-\}$$

IR Description of Framed BPS States: Say halo!

ζ near $\ell_{\gamma_h, u}$

A good description of some states in

$$\mathcal{H}^{\text{BPS}}_{L,\gamma}$$

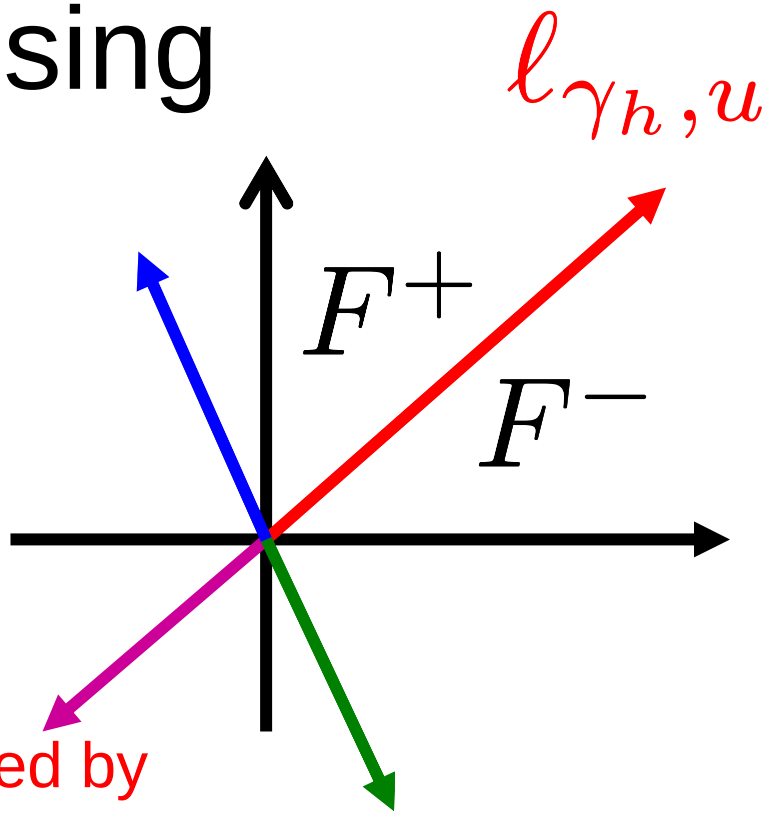


$$r_{\text{halo}} = \frac{\langle \gamma_c, \gamma_h \rangle}{2\text{Im}(Z_{\gamma_h}/\zeta)}$$

Wall-crossing

ζ crosses $\ell_{\gamma_h,u}$

We gain & lose halo Fock-spaces, exactly as in the derivation of the semi-primitive WCF!



Wall-crossing is elegantly summarized by introducing a generating function:

$$F = \sum_\gamma \overline{\underline{\Omega}}(L,\gamma;y) X_\gamma$$

How is F+ related to F- ?

Wall-crossing: Noncommutativity

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1 , \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}$$

$$F^{+} = S_{\gamma_h} F^{-} S_{\gamma_h}^{-1}$$

S_{γ_h} Product of quantum dilogs $\Phi(X_{\gamma_h})$

Wall-Crossing II

$$\Phi(X) = \prod_{k=1}^{\infty} (1 + y^{2k-1} X)$$

$$\Omega(\gamma_h; -y) = \sum_m a_m^{\gamma_h} y^m$$

$$S_{\gamma_h} = \prod_m \Phi((-y)^m X_{\gamma_h})^{a_m^{\gamma_h}}$$

Wall-Crossing III

This result implies the ``motivic wall crossing formula'' of Kontsevich & Soibelman

Our discussion is consistent with the form of the result as discussed in

Dimofte & Gukov; Cecotti & Vafa; Dimofte, Gukov & Soibelman

Relation to ``Darboux'' coordinates on Seiberg-Witten Moduli Spaces

Now we explain how line operators are related to an interesting collection of functions

$$\mathcal{X}_{\gamma}$$

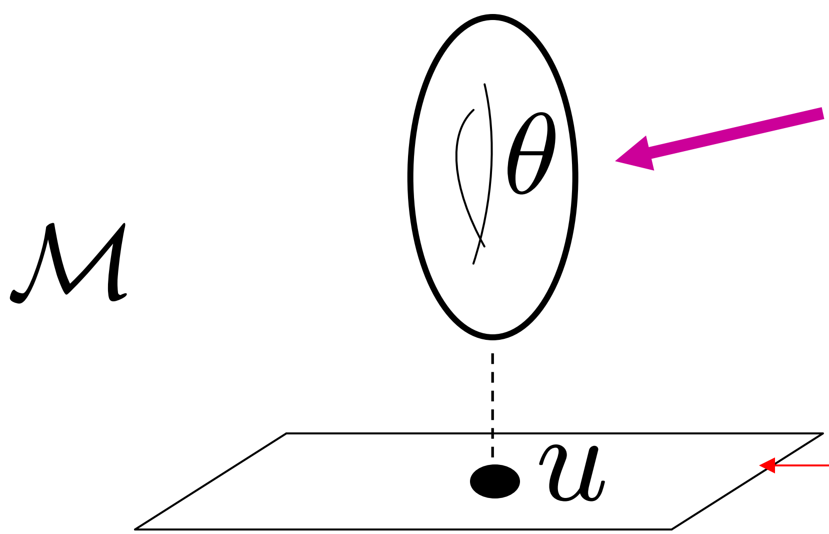
$\Rightarrow$ Lightning review of

Gaiotto, Moore, Neitzke, arXiv:0807.4723

Circle Compactification of N=2

N=2 Supersymmetric theory on $\mathbb{R}^3 \times S^1_R$

Low energies: Sigma model $\mathbb{R}^3 \to \mathcal{M}$



Torus of (elec, mag) Wilson lines

Coulomb branch

Susy $\Rightarrow$ $\mathcal{M}$ hyperkahler

``Darboux coordinates''

Giving a HK metric is equivalent to giving holomorphic symplectic structure on $\mathcal{M}^{\zeta} \quad \zeta \in \mathbb{P}^1$

$$\varpi_{\zeta} = \zeta^{-1}\omega_{+} + \omega_3 + \zeta\omega_{-}$$

ϖ_{ζ} Is determined from a collection of functions

$$\mathcal{X}_{\gamma} : \mathcal{M}^{\zeta} \times \mathbb{C}^{*} \to \mathbb{C} \qquad \gamma \in \Gamma$$

$$\mathcal{X}_{\gamma}\mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$$

$$\varpi_{\zeta} = \langle d\log\mathcal{X}_{\gamma}, d\log\mathcal{X}_{\gamma}\rangle$$

Constructing $\mathcal{X}_\gamma$

$$\mathcal{X}^{\rm sf}_\gamma := \exp\left[\frac{\pi R}{\zeta} Z_\gamma + i\gamma\cdot\theta + \pi R \zeta \bar{Z}_\gamma\right]$$

(Neitzke, Pioline, & Vandoren)

Solve a TBA-like integral equation

$$\log \mathcal{X}_\gamma = \log \mathcal{X}^{\rm sf}_\gamma +$$

$$+ \sum_{\gamma'} \Omega(\gamma') K_{\gamma,\gamma'} * \log(1 + \mathcal{X}_{\gamma'})$$

IR Line Operator Expansion

L_ζ wraps circle in $\mathbb{R}^3 \times S^1_R$

$$\langle L_\zeta \rangle = \sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma) \mathcal{X}_\gamma$$

Holomorphic on $\mathcal{M}^\zeta$

$$\langle L_\zeta \rangle = \mathrm{Tr}_{\mathcal{H}_{L_\zeta}} (-1)^F e^{-2\pi R H} e^{i\theta \cdot \mathcal{Q}}$$

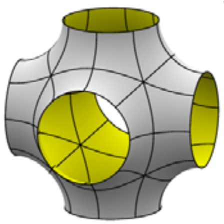
$$\xrightarrow[R \to \infty]{} \sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma) e^{2\pi R \mathrm{Re}(Z_\gamma/\zeta) + i\gamma\cdot\theta}$$

$\langle L_\zeta \rangle$ Has no wall-crossing!

Six-dimensional (2,0) theory

$d = 6\ (2,0)\ \mathfrak{g}\text{-theory}$

+ topological twist

C: 

d=4 $\mathcal{N}$ =2 theory $T_{g,n}[\mathfrak{g}]$

Generalizing a construction of Witten 98, GMN studied these theories in order to construct Darboux coordinates, and found an algorithm for computing the BPS spectrum for $T_{g,n}[A_1]$ theories.

They have attracted a lot of attention following the discovery by Gaiotto, arXiv:0904.2715, that they can be described as generalized quiver gauge theories.

6D to 3D

C: Genus g surface with n punctures

$$
\begin{tikzcd}[column sep=small]
 & \boxed{d = 6\ (2,0)\ \mathfrak{g}\text{-theory}[\mathbb{R}^3 \times S^1_R \times C]} \arrow[dl, "C"'] \arrow[dr, "S^1_R"] & \\
d = 4\text{: } T_{g,n}[\mathfrak{g}] \arrow[dr, "S^1_R"'] & & d = 5\text{: } \mathfrak{g}\text{-SYM} \arrow[dl, "C"] \\
 & \mathbb{R}^3 \to \mathcal{M} &
\end{tikzcd}
$$

Sigma model

Hitchin = Seiberg-Witten

$$\mathcal{M}: \left.\begin{aligned} F + R^2[\varphi, \bar{\varphi}] = 0 \\ \bar{\partial}_A \varphi = 0 \end{aligned}\right\} \mathfrak{g}$$

Hitchin system on C

Spectral curve in T^*C $\quad \Sigma : \det(\lambda - \varphi) = 0$

Σ Seiberg-Witten curve

λ Seiberg-Witten differential

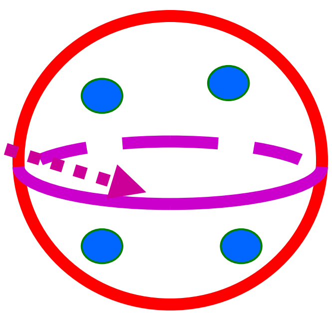
Flat Connections

$$\text{For } \zeta \neq 0, \infty$$

$$\mathcal{M}^{\zeta} = \text{ Moduli of flat } \mathfrak{g}_{c} \text{ connections}$$

$$\mathcal{A}_{\zeta} = R\zeta^{-1}\varphi + A + R\zeta\bar{\varphi}$$

$$\text{For } \wp \subset C:$$



$$\mathrm{Hol}(\mathcal{R}, \wp) := \mathrm{Tr}_{\mathcal{R}} P \exp \oint_{\wp} \mathcal{A}_{\zeta}$$

will be an important holomorphic function for us.

Surface to Line Operators

$$\begin{tikzcd}
& \langle \mathbb{S}(\mathcal{R}, S^1_R \times \wp) \rangle \arrow[dl, "C"'] \arrow[dr, "S^1_R"] & \\
\langle L_\zeta(\mathcal{R}, \wp) \rangle & = & \mathrm{Tr}_{\mathcal{R}} P \exp \oint_\wp \mathcal{A}_\zeta
\end{tikzcd}$$

L_ζ labeled by $\wp \subset C$ and $\mathcal{R}$:

Consistent with Drukker,Morrison, Okuda

How to compute $\underline{\overline{\Omega}}$

$$\langle L_\zeta \rangle = \sum_\gamma \underline{\overline{\Omega}}(L_\zeta, \gamma) \mathcal{X}_\gamma$$

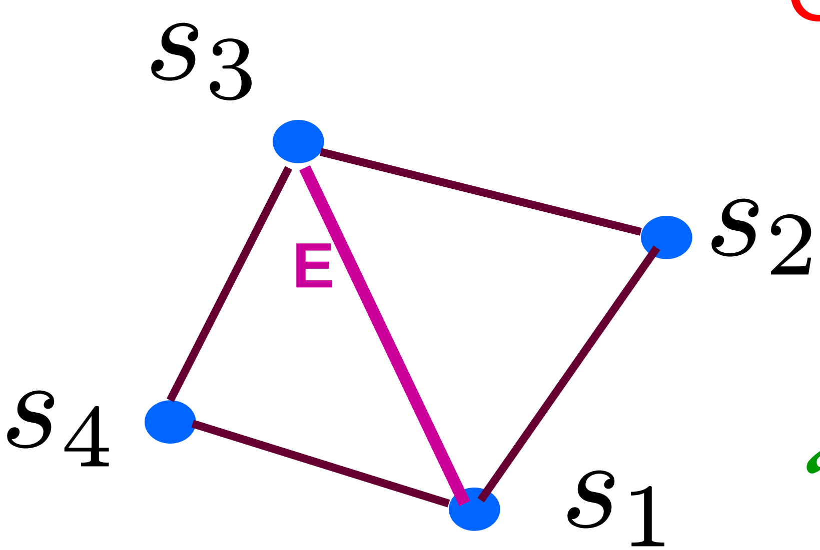
$$\langle L_\zeta(\mathcal{R}, \wp) \rangle = \mathrm{Hol}(\mathcal{R}, \wp)$$

For $T_{g,n}[A_1]$ theories:

1. Expand $Hol(\mathcal{R}, \wp)$ using Fock-Goncharov coordinates $\mathcal{X}_E$
2. Write $\mathcal{X}_\gamma$ in terms of $\mathcal{X}_E$

Fock-Goncharov Coordinates

Choose a triangulation of C with vertices at the punctures: $\Rightarrow$ Coordinates $\mathcal{X}_E$ on $\mathcal{M}^\zeta$



Choose flat sections s_i

$$(d + \mathcal{A}_\zeta)s_i = 0$$

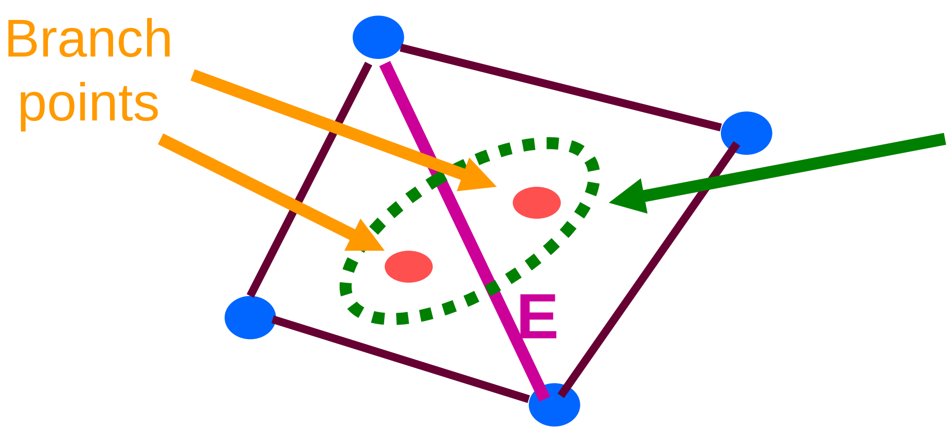
$$\mathcal{X}_E := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}$$

Fock-Goncharov gives Darboux

$$2:1 \quad \Sigma \to C \qquad \Gamma = H_1(\Sigma;\mathbb{Z})^-$$

Use canonical triangulation of C:

from integral curves of λ



γ_E (nice!) basis for Γ

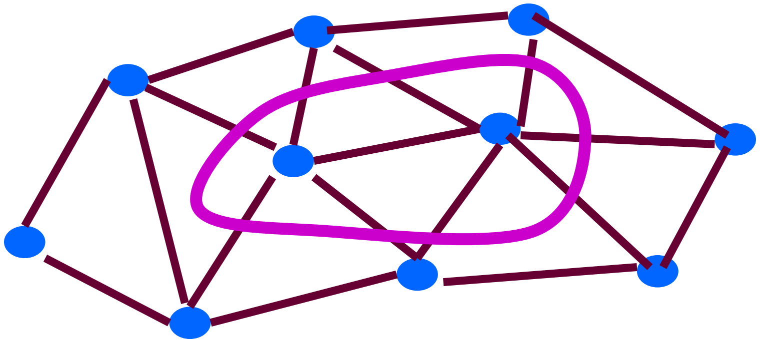
$$\mathcal{X}_{\gamma_E} := \mathcal{X}_E$$

Computing the Holonomy

$$\mathrm{Hol}(\mathcal{R}, \wp) = \sum_\gamma c_\gamma \mathcal{X}_\gamma$$

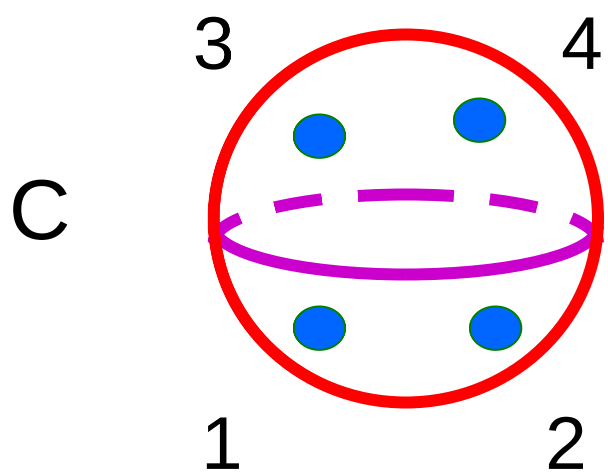
Computable via a simple traffic rule algorithm

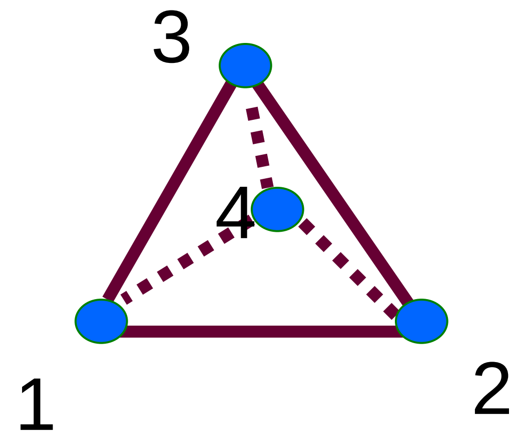
$$R = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad M[x] = \frac{1}{\sqrt{x}} \begin{pmatrix} 0 & -1 \\ -x & 0 \end{pmatrix}$$



$\Rightarrow$ $\overline{\underline{\Omega}}(L_\zeta(\mathcal{R}, \wp))$ computable for $T_{g,n}[A_1]$

Example: SU(2) N_f=4





$$\frac{1+X_{13}+X_{24}+X_{13}X_{24}+X_{13}X_{14}X_{24}+X_{13}X_{23}X_{24}+X_{13}X_{14}X_{23}X_{24}}{\sqrt{X_{13}X_{14}X_{23}X_{24}}}$$

Would be nontrivial to compute from the geometry of monopole moduli spaces!

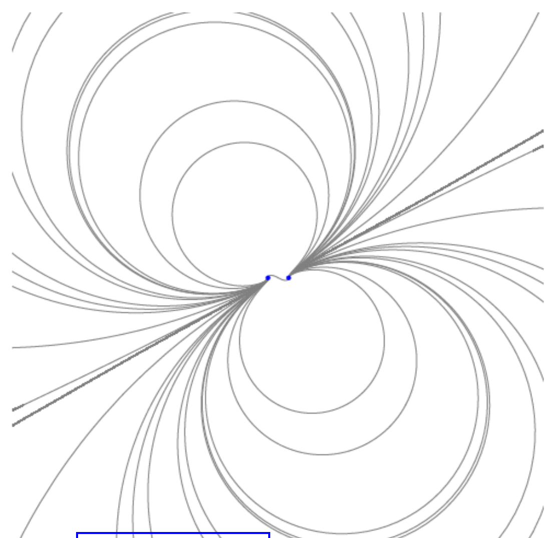
Asymptotically Free Theories:

$T_{g,n}[A_1]$ are perturbations of conformal field theories

Reach AF theories by decoupling HM's: Send masses to infinity

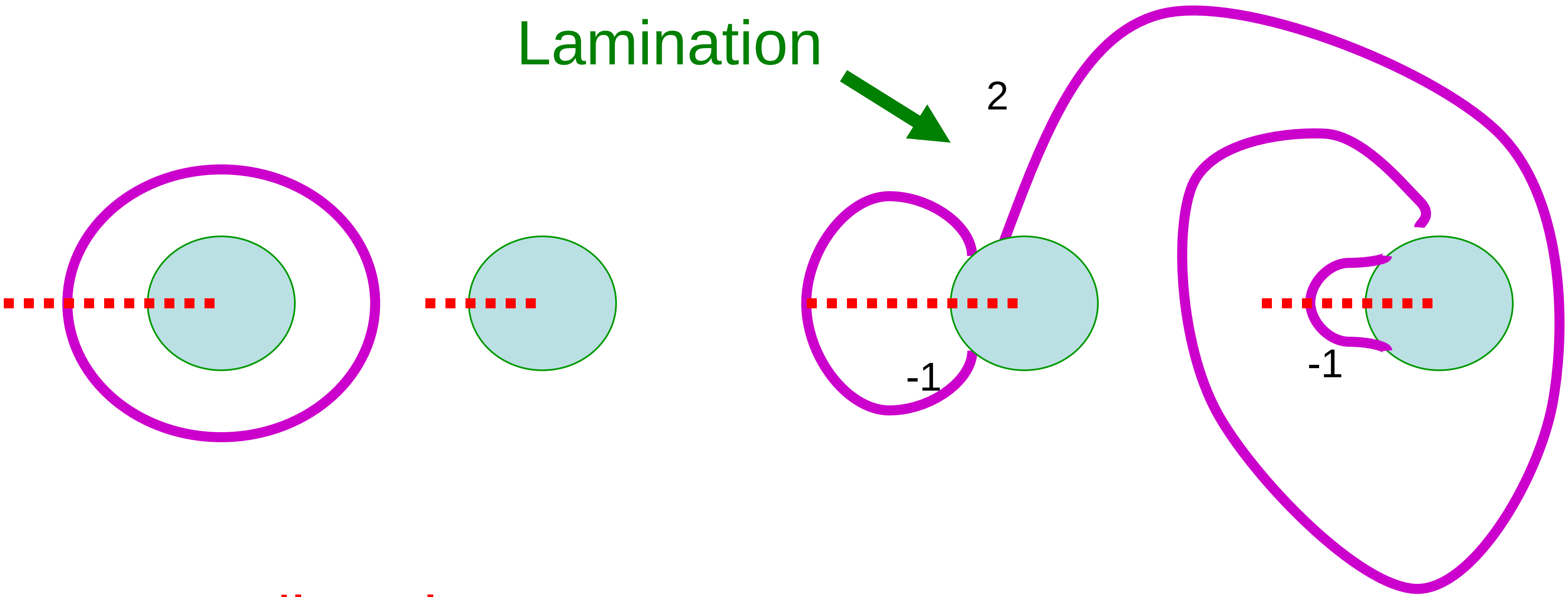
Geometrically:

Collide RSP's to get ISP's



UV labeling of line operators: Closed Curves $\Rightarrow$ Laminations

Example: SU(2) N_f=0



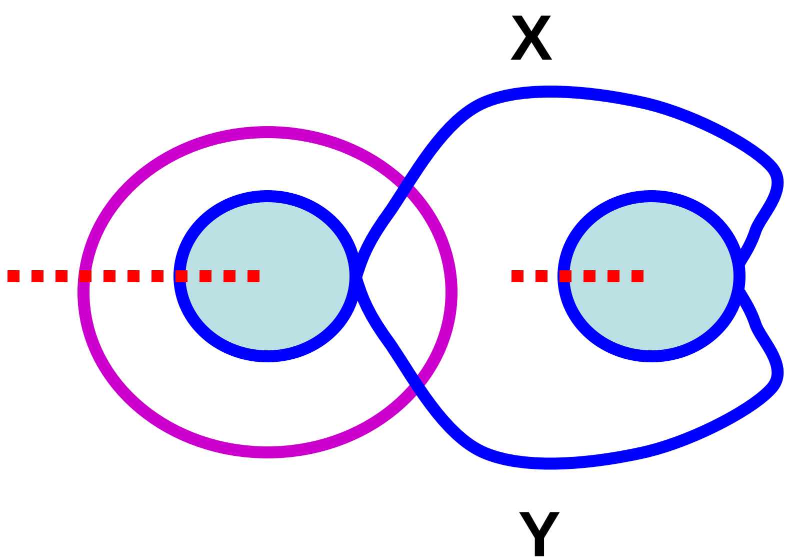
Wilson loop:

Charge (0,1)

't Hooft-Wilson loops:

Charge (1,2n)

N_f=0 Framed BPS Degeneracies



$$\Gamma = \mathbb{Z}^2$$

$$\mathcal{X}_\gamma = X^m Y^n$$

$$\langle W_{\mathbf{2}} \rangle = \underbrace{(XY)^{1/2} + (XY)^{-1/2}}_{\text{Naive}} + \sqrt{\frac{X}{Y}}$$

Surprise !

Many elaborate results for lamination vevs

Quantum Holonomy

Classically, we have interpreted the framed BPS indices as traces of holonomy:

$$\mathrm{Hol}(\mathcal{R},\wp) = \sum_\gamma \underline{\overline{\Omega}}(L_\zeta(\wp),\gamma)\mathcal{X}_\gamma$$

What about the PSC?

Combining with results of Teschner on quantization of Teichmuller space shows that

$$\mathcal{O}(\mathrm{Hol}(\mathcal{R},\wp)) \sim \sum_\gamma \underline{\overline{\Omega}}(L_\zeta(\wp),\gamma;y)X_\gamma$$

Satisfy the same operator algebra.

Looking Ahead…

There's been lots of progress on $\mathcal{N}=2$, and there's lots more to do: this will surely keep us focused in the near future…

Litmus test: There is no effective algorithm for computing the BPS spectrum of an arbitrary N=2 FT or string cpct.


M5