

2d Categorical Wall-Crossing With Twisted Masses, And An Application To Knot Invariants

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Part of this talk is review of old things

Witten, 1982; Cecotti & Vafa, 1992

Some foundational material: D. Gaiotto, G. Moore, & E. Witten 2015 (GMW)

**New things are work with
AHSAN KHAN**

arXiv:2010.11837

arXiv:21???.?????



+ comment resulting from
discussions with



With nontrivial input from Andy and Tudor



Supersymmetric Quantum Mechanics And Homological algebra

1

2D $N=(2,2)$ Landau-Ginzburg Models

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Thimble Branes & Their Local Operators

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Categorical Wall-Crossing

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Generalization To Twisted Masses

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Relation To 3d Indices

SQM & Morse-Novikov Theory (Witten: 1982)

M : Riemannian; α : Closed 1-form

Locally $\alpha = dh$ with $h: M \rightarrow \mathbb{R}$ the traditional superpotential.

But h need not be single-valued: α need not be exact.

Call α the "super-one-form"

SQM: $\phi: \mathbb{R}_t \rightarrow M$

$$L = g_{IJ}(\phi) \dot{\phi}^I \dot{\phi}^J - \|\alpha\|^2 + \dots$$

SQM & Morse-Novikov Theory (Witten: 1982)

SQM: $\phi: \mathbb{R}_t \rightarrow M$

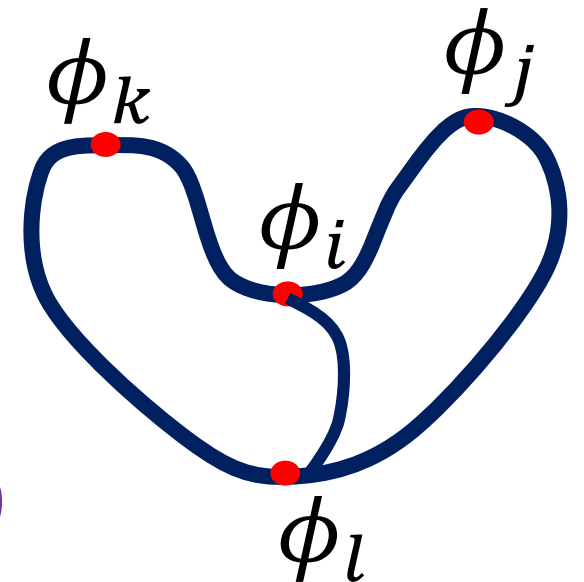
$$L = g_{IJ}(\phi) \dot{\phi}^I \dot{\phi}^J - \|\alpha\|^2 + \dots$$

Classical vacua: $\alpha(\phi_i) = 0$ "critical points"

"Massive" $D_I \alpha_J|_{\phi_i}$ is invertible

➔ Approximate quantum vacua: $\Psi(\phi_i)$

Fermion number: $F(\Psi(\phi_i)) = \frac{1}{2} (d_-(\phi_i) - d_+(\phi_i))$



Instantons & MSW Complex

The approximate vacua are not exact because of instanton effects.

Instanton equation: $\frac{d\phi^I}{d\tau} = g^{IJ}(\phi)\alpha_J(\phi)$

Use instantons to define an operator Q
on approximate ground states

$$Q\Psi(\phi_i) := \sum_{F_j=F_i+1} n_{j,i} \Psi(\phi_j)$$

Broken flows: $\sum_{F_r=F_q+1=F_p+2} n_{pq}n_{qr} = 0$

$$\Rightarrow Q^2 = 0$$

MSW Chain Complex

$$\mathcal{C} := (V, F, Q)$$

SQM: $V \subset \mathcal{H}$: The span of the approximate ground states $\Psi(\phi_i)$

F : Fermion number
(valued in a \mathbb{Z} -torsor)

Q : Susy operator

$$[F, Q] = Q$$

$$Q^2 = 0$$

MSW complex: $\text{MSW}(M, g_{IJ}, \alpha_I)$

Exact ground states $\cong H^*(V, Q) \cong H^*(M; d + \alpha)$

Homotopies Of Metric And Super-one-form

Now consider a continuous family: $(g_{IJ}(\phi; s), \alpha_I(\phi; s))$ $s_1 \leq s \leq s_2$

How does the MSW complex change?

Define $\mathcal{U}: MSW(s_1) \rightarrow MSW(s_2)$

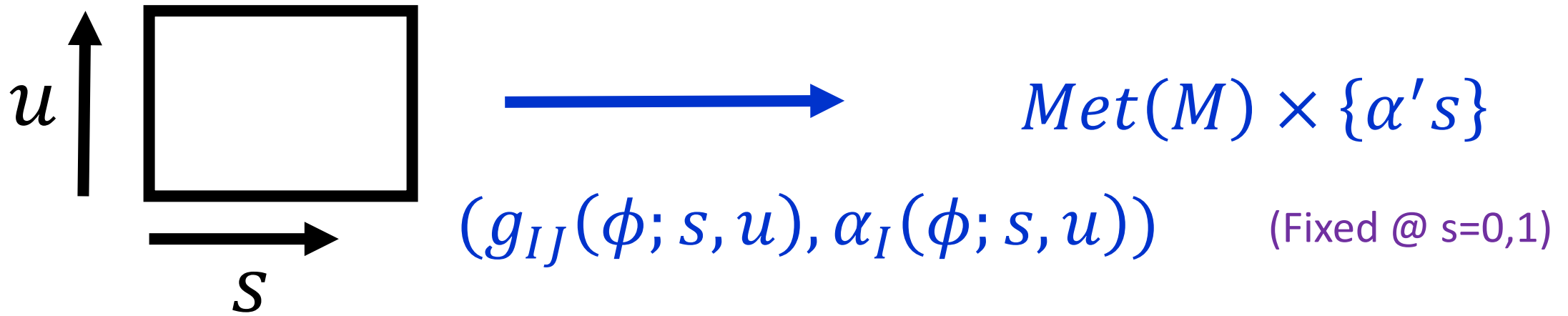
$$\frac{d\phi^I}{ds} = g^{IJ}(\phi; s)\alpha_J(\phi; s) \quad \mathcal{U} \Psi(p; s_1) = \sum_{F_q = F_p} n_{p,q} \Psi(q; s_2)$$

Claim: $\mathcal{U} F = F \mathcal{U} \quad \mathcal{U} Q_1 = Q_2 \mathcal{U}$

Under continuous deformation of
metric and super-one-form
the MSW complex changes by
a chain map.

Actually, it is a very special kind of chain map:
A homotopy equivalence of chain complexes.

Homotopies Of Paths \Rightarrow Homotopy Of Chain Maps



$u = 0$ gives a chain map $\mathcal{U}_0 : (V, F, Q)_1 \rightarrow (V, F, Q)_2$

$u = 1$ gives a chain map $\mathcal{U}_1 : (V, F, Q)_1 \rightarrow (V, F, Q)_2$

$$\mathcal{U}_0 - \mathcal{U}_1 = Q_2 E + E Q_1 \quad \frac{d\phi^I}{ds} = g^{IJ}(\phi; s, u) \alpha_J(\phi; s, u)$$

$$E \Psi(p; s = 0, u = 0) = \sum_{F_q = F_p - 1} n_{p,q} \Psi(q; s = 1, u = 1)$$

Definition: Homotopies Of Chain Maps

Two chain maps $f_0, f_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ are homotopic if

There is a Fermion number -1 map $E: V_1 \rightarrow V_2$

$$f_0 - f_1 = Q_2 E + E Q_1$$

If there are
chain maps....

$$f: \mathcal{C}_1 \rightarrow \mathcal{C}_2 \quad g: \mathcal{C}_2 \rightarrow \mathcal{C}_1$$

$$g \circ f \sim Id_{\mathcal{C}_1} \quad f \circ g \sim Id_{\mathcal{C}_2}$$

Then the chain complexes \mathcal{C}_1 and \mathcal{C}_2 are homotopy equivalent.

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Landau-Ginzburg Models $LG(X, \alpha)$

$$X, g_{I\bar{J}}: \text{Kähler} \quad \alpha \in \Omega^{1,0}(X) \ , \ \bar{\partial}\alpha = 0$$

Locally $\alpha = \partial W$ with $W: X \rightarrow \mathbb{C}$, but we will be considering multi-valued W

$$S = \int_{\mathbb{R}} dt \int_D dx \left(g_{I\bar{J}}(\phi(x, t)) \partial_\mu \phi^I \partial^\mu \phi^{\bar{J}} - \|\alpha(\phi)\|^2 + \dots \right)$$

Poincare invariant vacua for $D = \mathbb{R}$:

$$\mathbb{V} = \{ \phi_i \mid \alpha(\phi_i) = 0 \}$$

Branes

$$D = [x_0, \infty)$$

2d LG Model As 1d SQM

Consider SQM with target: $\mathcal{X} = \text{Map}(\phi: D \rightarrow X)$

$$\|\delta\phi\|^2 = \int_D g_{I\bar{J}}(\phi(x)) \delta\phi^I \delta\phi^{\bar{J}}$$

$\alpha \in \Omega^{1,0}(X)$ induces a super-one-form $\tilde{\alpha}$ on \mathcal{X}

$$\tilde{\alpha}[\phi] = \int_D [\phi^*(\omega) - \text{Re}(\zeta^{-1} \alpha_I(\phi) \delta\phi^I) dx]$$

$$\text{SQM}(\mathcal{X}, \tilde{\alpha}) = \text{LG}(X, \alpha)$$

Superpotential With Twisted Masses

Usual discussion: $\alpha = \partial W$

with $W: X \rightarrow \mathbb{C}$ holomorphic and Morse

If α has nonzero periods there is no single-valued superpotential

“twisted masses”

\exists Minimal Abelian cover $\pi: \hat{X} \rightarrow X$ so that $\pi^*(\alpha) = \partial \hat{W}$

Γ : Free Abelian Deck group $\subset H_1(X; \mathbb{Z})$

It is often convenient to consider

$$LG(\hat{X}, \hat{\alpha} = \partial \hat{W})$$

and work equivariantly wrt Γ

$$\text{Vacua of } LG(\hat{X}, \hat{\alpha} = \partial \hat{W}) \quad : \quad \hat{\mathbb{V}} = \{ \hat{\phi}_a \mid d\hat{W}(\hat{\phi}_a) = 0 \}$$

Abbreviate vacua $\hat{\phi}_a, \hat{\phi}_b, \dots$ simply by a, b, \dots

Write free Γ -action on $\hat{\mathbb{V}}$: $a \rightarrow a + \gamma$

$$\hat{W}_{a+\gamma} = \hat{W}_a + \oint_{\gamma} \alpha$$

Example 1: Mirror Of The Free Chiral

$$X = \mathbb{C}^* \quad \alpha = \left(\frac{m}{\phi} - 1 \right) d\phi \quad \mathbb{V} = \{ \phi_0 = m \}$$

$$\pi: \hat{X} = \mathbb{C} \rightarrow X = \mathbb{C}^* \quad \pi: \hat{\phi} \rightarrow \phi = \exp(\hat{\phi}) \quad \Gamma \cong \mathbb{Z}$$

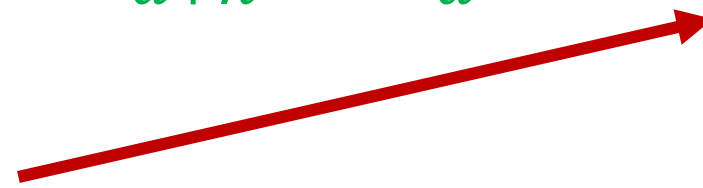
$$\hat{\alpha} = d\hat{W} = d(m\hat{\phi} - e^{\hat{\phi}}) \leftarrow \alpha = \left(\frac{m}{\phi} - 1 \right) d\phi$$

$$\hat{\mathbb{V}} = \{ \hat{\phi}_a = \log m + 2\pi i a \mid a \in \mathbb{Z} \}$$

$$\hat{W}_a = m \log m + 2\pi i a m$$

$$\hat{W}_{a+n} = \hat{W}_a + 2\pi i m n$$

Twisted mass



Other Examples

Mirror of $\mathbb{C}\mathbb{P}^1$: $\alpha = \left(\frac{t}{\phi^2} + \frac{m}{\phi} + t \right) d\phi \quad \phi \in X = \mathbb{C}^*$

Discussed in GMW framework in Galakhov (2021) and Khan-Moore, to appear

There are two vacua ϕ_i, ϕ_j and rank one deck group $\Gamma \cong \mathbb{Z}$

LG models for knot homology

[Gaiotto-Witten; Galakhov-Moore; Aganagic]

Chern-Simons-Landau-Ginzburg

$G_{\mathbb{C}}$: Complex Lie group

M_3 : Riemannian 3-fold \Rightarrow LG model $\text{CSLG}[G_{\mathbb{C}}, M_3]$

$X = \{ \text{Complex } G_{\mathbb{C}} \text{ - connections on a 3 - manifold } M_3 \}$

$$\alpha = \int_{M_3} \text{Tr } \mathcal{F}^2 \quad \text{``} = d \text{CS}(\mathcal{A}) \text{''}$$

Vacua on $D = \mathbb{R}$: Flat $G_{\mathbb{C}}$ - connections

Morse Theory Flows In LG Language

$SQM(\mathcal{X})$ vacua:

$$\tilde{\alpha} = 0$$



$$\frac{\partial \phi^I}{\partial x} = i \zeta g^{I\bar{J}} \bar{\alpha}_{\bar{J}}(\phi)$$

We call this the ζ -soliton equation


$$\frac{\partial \phi}{\partial \tau} = \tilde{\alpha}$$



$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial \tau} \right) \phi^I = i \zeta g^{I\bar{J}} \bar{\alpha}_{\bar{J}}(\phi)$$

We call this the ζ -instanton equation

MSW Complex For $SQM(\mathcal{X}, \tilde{\alpha})$

$$\phi(x) \rightarrow \phi_i \qquad \qquad \qquad \phi(x) \rightarrow \phi_j$$


$$R_{ij} = (\text{Span}\{\Psi[\phi_{ij}(x)]\}, F_{ij}, Q_{ij})$$

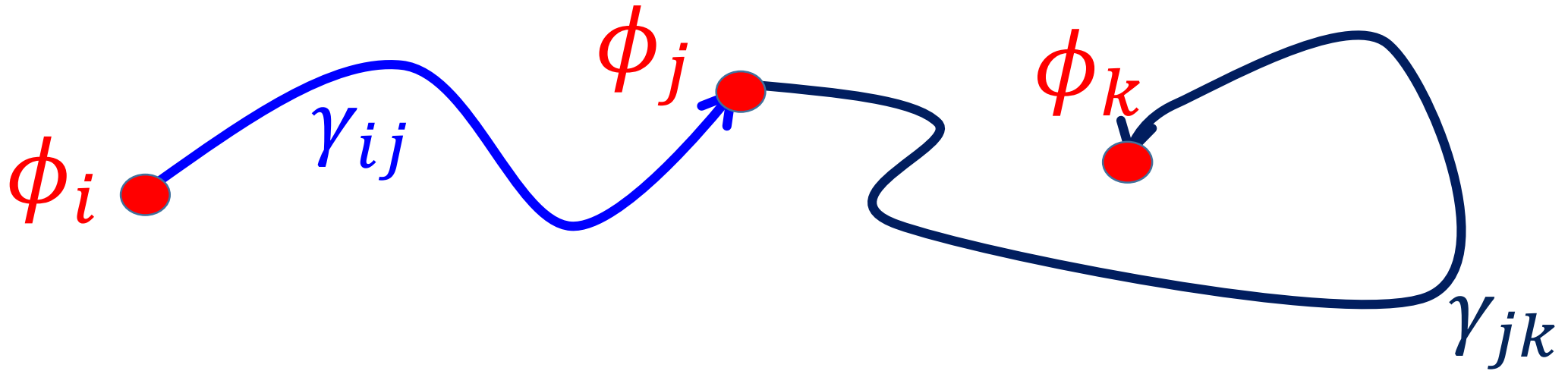
$$F_{ij} \Psi_{ij} = \eta \left(D_{\phi_{ij}} \right) \Psi_{ij} \qquad Q_{ij} : \text{Count } \zeta \text{ -instantons}$$

$$\text{``Flavor Charge''} : \quad [\phi_{ij}(\mathbb{R})] = \gamma_{ij} \in \Gamma_{ij}$$

Γ_{ij} = paths in X from ϕ_i to ϕ_j --- up to homology.

Adding Charges

Composition of curves defines $\gamma_{ij} + \gamma_{jk} \in \Gamma_{ik}$



Abelian group structure on $\Gamma_{ii} \cong \Gamma$.

Γ_{ij} is a Γ -torsor

R_{ij} is graded by Γ -torsor Γ_{ij} $R_{ij} = \bigoplus_{\gamma_{ij} \in \Gamma_{ij}} R_{\gamma_{ij}}$

“BPS index” $\mu_{\gamma_{ij}} := \text{Tr}_{R_{\gamma_{ij}}} e^{i\pi F_{ij}}$

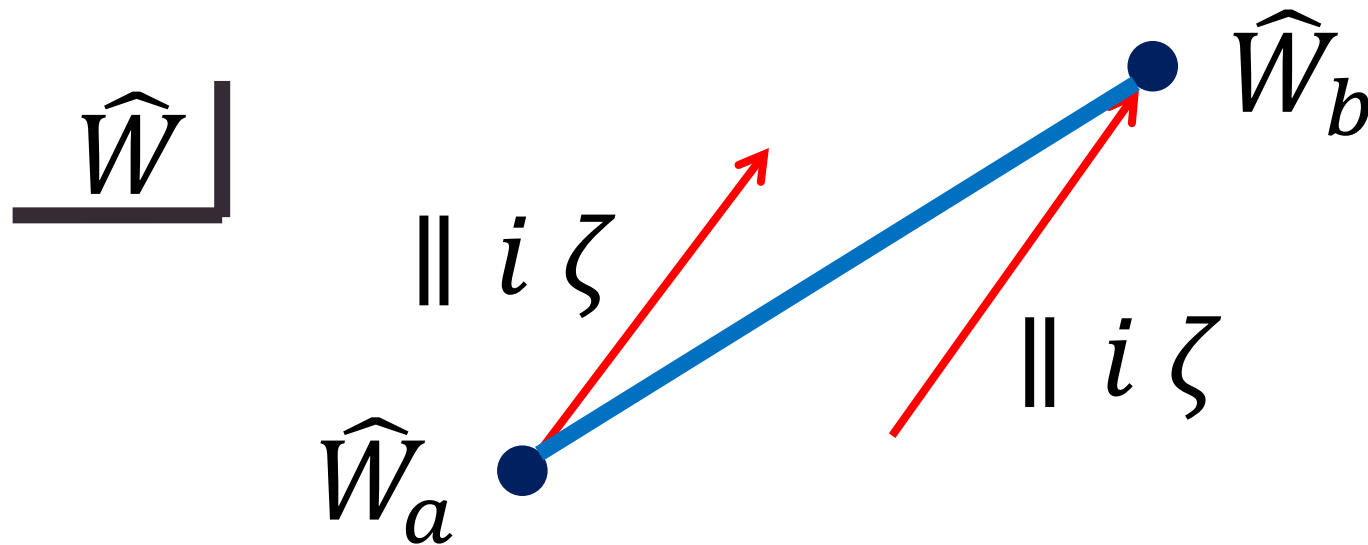
Central charge: $Z_{\gamma_{ij}} = \oint_{\gamma_{ij}} \alpha$

Generalizes the standard $Z_{ij} = W_i - W_j$

“Twisted mass property” $Z_{\gamma_{ij} + \gamma} = Z_{\gamma_{ij}} + Z_{\gamma}$

Working On The Cover

$$\frac{\partial \hat{\phi}^I}{\partial x} = i \zeta g^{I\bar{J}} \frac{\partial \widehat{W}}{\partial \hat{\phi}^{\bar{J}}}$$



$R_{\gamma_{ij}}$ is only nonzero for $i \zeta \parallel$ to the phase of $Z_{\gamma_{ij}}$

Periodic Solitons

Qualitatively new feature: If $b = a + \gamma$

$$\phi_i = \pi(\hat{\phi}_b) = \pi(\hat{\phi}_a)$$

$\phi(x) \rightarrow \phi_i$

$\phi(x) \rightarrow \phi_i$



Without gradient flow we can have nontrivial solutions!

$$Z_{\gamma ii} = \int_{\gamma} \alpha$$

$\Gamma_{ii} \cong \Gamma \Rightarrow R_{ii}$ is a $\mathbb{Z}_F \times \Gamma$ -graded complex

Main new ingredient in categorified
wall-crossing with twisted masses
involves Fock spaces constructed from R_{ii}

Similar to the “halo picture” of 4d KSWCF

[Denef & Moore; GMN ; Andriyash, Denef, Jafferis, Moore]

Wall-Crossing When $\alpha = \partial W$

$$\left(g_{I\bar{J}}(\phi; s), W(\phi; s) \right) \quad 0 \leq s \leq 1$$

$\mu_{ij}(s)$ is only piecewise constant: CFIV, CV 1991, 1992

Very different from the finite-dimensional situation!

CVWCF: Tells how $\mu_{ij}(s)$ jump.

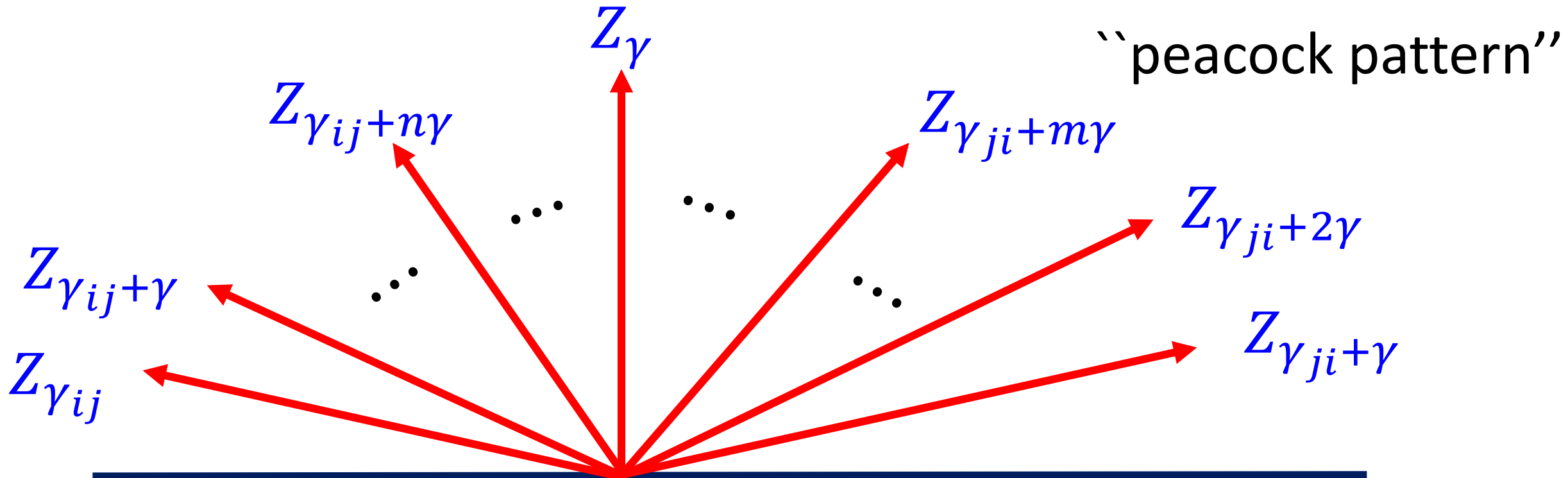
Categorified CVWCF: Describe how the homotopy equivalence class of R_{ij} jumps.

Wall-Crossing With Twisted Masses

Qualitatively-different because we can have new walls:

$$\text{Im} \left(Z_{\gamma_{ij}} \bar{Z}_{\gamma} \right) = 0 \quad i \neq j, \quad \gamma \in \Gamma$$

Example: Case of just two vacua: ϕ_i & ϕ_j & $\Gamma = \mathbb{Z}\gamma$ is rank one.



Wall-Crossing With Twisted Masses: Vacuum Groupoid Algebra

Vacuum Category: Objects = $\{\phi_i\}$ Morphisms = $\{\gamma_{ij}\}$

“Vacuum Groupoid Algebra” :

For each $\gamma_{ij} \in \Gamma_{ij}$ introduce a variable $x_{\gamma_{ij}}$

$$x_{\gamma_{ij}} x_{\gamma_{kl}} = \delta_{jk} x_{\gamma_{ij} + \gamma_{jk}}$$

2d – 4d Spectrum Generator

$$\forall i \neq j \ \& \ \gamma_{ij} \in \Gamma_{ij} \quad S_{\gamma_{ij}} := 1 + \mu_{\gamma_{ij}} x_{\gamma_{ij}}$$

$$\forall \gamma \in \Gamma \quad K_{\gamma} := \sum_i \prod_k (1 - x_{\gamma})^{-\mu_{k\gamma ii}} x_{u_i} \quad \begin{array}{l} u_i \in \Gamma_{ii} \\ \text{Additive identity} \end{array}$$

For any half-plane $\mathbb{H} \subset \mathbb{C}$

$$S(\mathbb{H}) =: \prod_{Z_{\gamma_{ij}}, Z_{\gamma} \in \mathbb{H}} S_{\gamma_{ij}} K_{\gamma} \quad : \quad \text{Phase-ordered product}$$

2d-4d Wall-Crossing Formula

[Kontsevich,Soibelman 2008; Gaiotto,Moore,Neitzke 2010]

Wall-crossing statement (GMN,KS):

$S(\mathbb{H})$ is invariant

provided no BPS rays enter/leave the half-plane \mathbb{H}

Although there is no 4d here, the mathematics is formally similar.

Solution To 2d4d Wall-Crossing: Two Vacua

Example: Case of just two vacua: ϕ_i & ϕ_j & $\Gamma = \mathbb{Z}\gamma$ is rank one.

Go through wall $Im\left(Z_{\gamma ij} \bar{Z}_{\gamma}\right) = 0$

$$F_i = \prod_{k \geq 1} (1 - x_{\gamma}^k)^{-\mu_{k\gamma ii}}$$

$$F'_i = F_i + A_{ij} F_j A_{ji}$$

$$A_{ij} = \sum_{k \geq 0} \mu_{\gamma ij + k\gamma} x_{\gamma}^k$$

$$A'_{ij} = (F_i + A_{ij} F_j A_{ji})^{-1} A_{ij} F_j$$

$$A_{ji} = \sum_{k \geq 0} \mu_{\gamma ji + k\gamma} x_{\gamma}^k$$

$$A'_{ji} = (F_i + A_{ij} F_j A_{ji})^{-1} F_j A_{ji}$$

$$F_j = \prod_{k \geq 1} (1 - x_{\gamma}^k)^{-\mu_{k\gamma jj}}$$

$$F'_j = (F_j + A_{ij} F_j A_{ji})^{-1} F_i F_j$$

We will give a categorical analog

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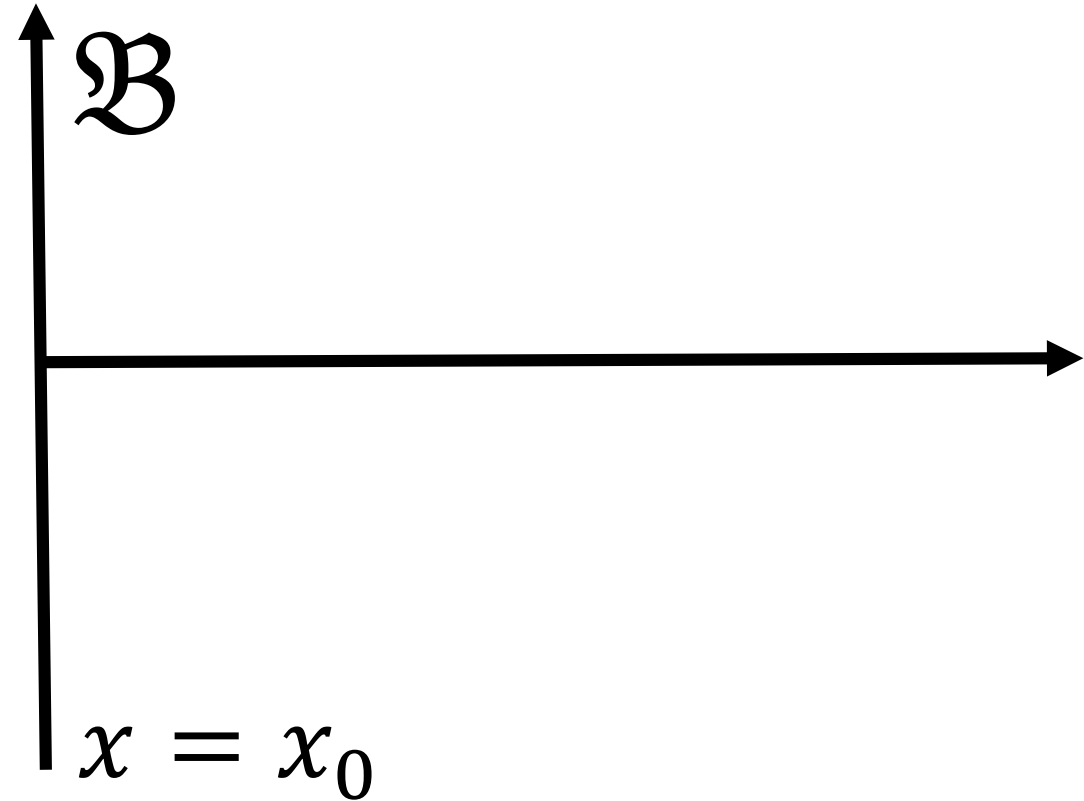
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Relation To 3d Indices

Branes

Use the (A_∞ algebra/category of) Branes.

The homotopy class of
the category of branes
is invariant.



Lefschetz Thimbles

Choose a half plane \mathbb{H} .

For each vacuum ϕ_i there is a canonical brane \mathfrak{T}_i

Consider the possible values of ϕ at x_0 so there is a solution of

$$\mathcal{L}_i^{right}(\zeta) \subset X \quad \frac{\partial \phi^I}{\partial x} = i \zeta g^{I\bar{J}} \bar{\alpha}_{\bar{J}} \quad \begin{array}{l} \phi(x) \rightarrow \phi_i \\ x \rightarrow +\infty \end{array}$$

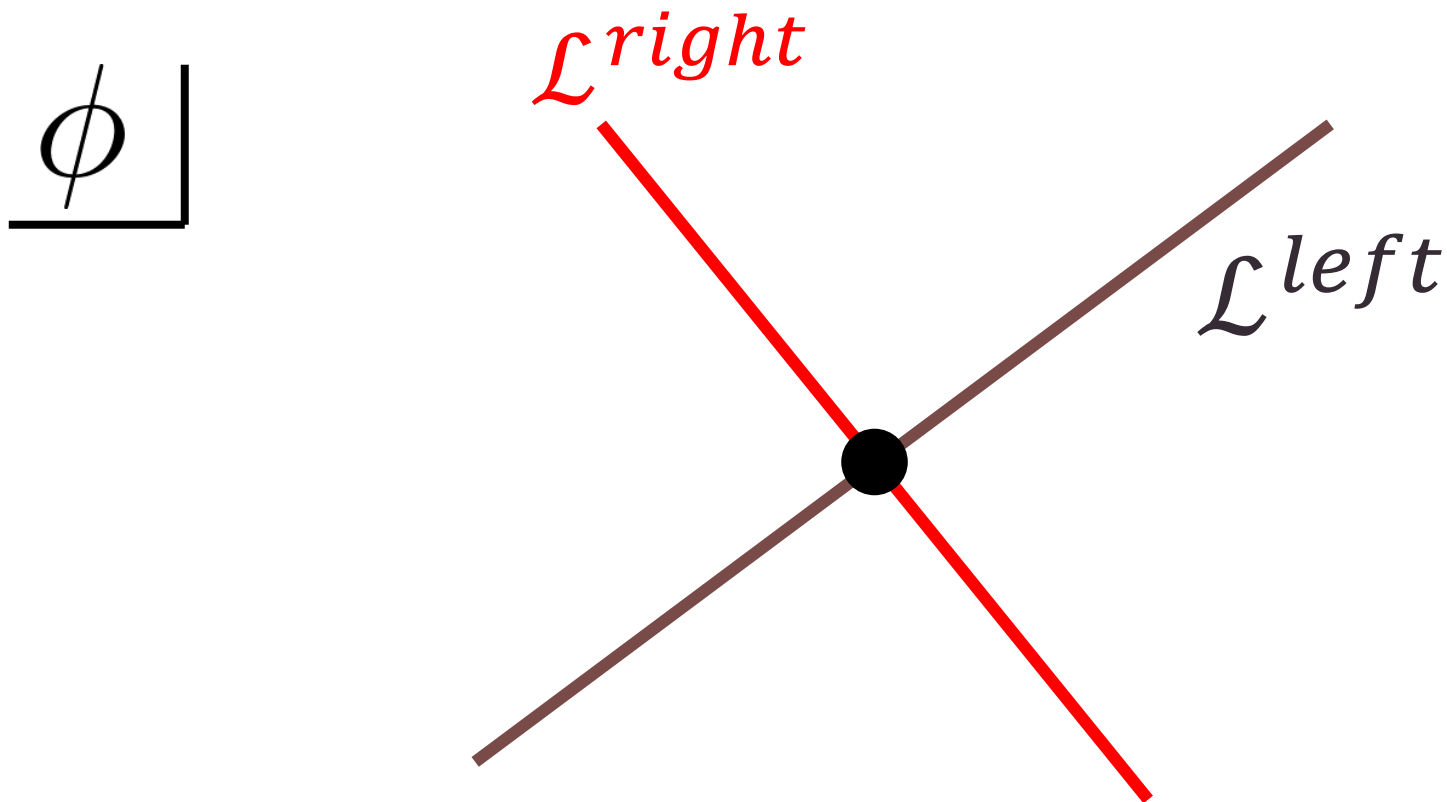
$\mathcal{L}_i^{right}(\zeta) \subset X$ are Lagrangian subspaces and provide nice half-susy bc's. [Hori-Iqbal-Vafa]

Preserves $Q_- - \zeta^{-1} \overline{Q_+}$

Example 1 of Lefschetz Thimbles

$$W = \frac{1}{2} \phi^2 \quad \frac{d\phi}{dx} = i \zeta \bar{\phi} \quad \longrightarrow \quad \phi(x) = c_0 \sqrt{\pm i \zeta} e^{\pm x}$$

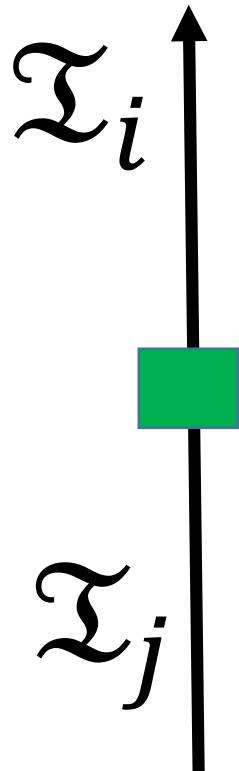
$c_0 \in \mathbb{R}$



Boundary Condition-Changing Local Operators

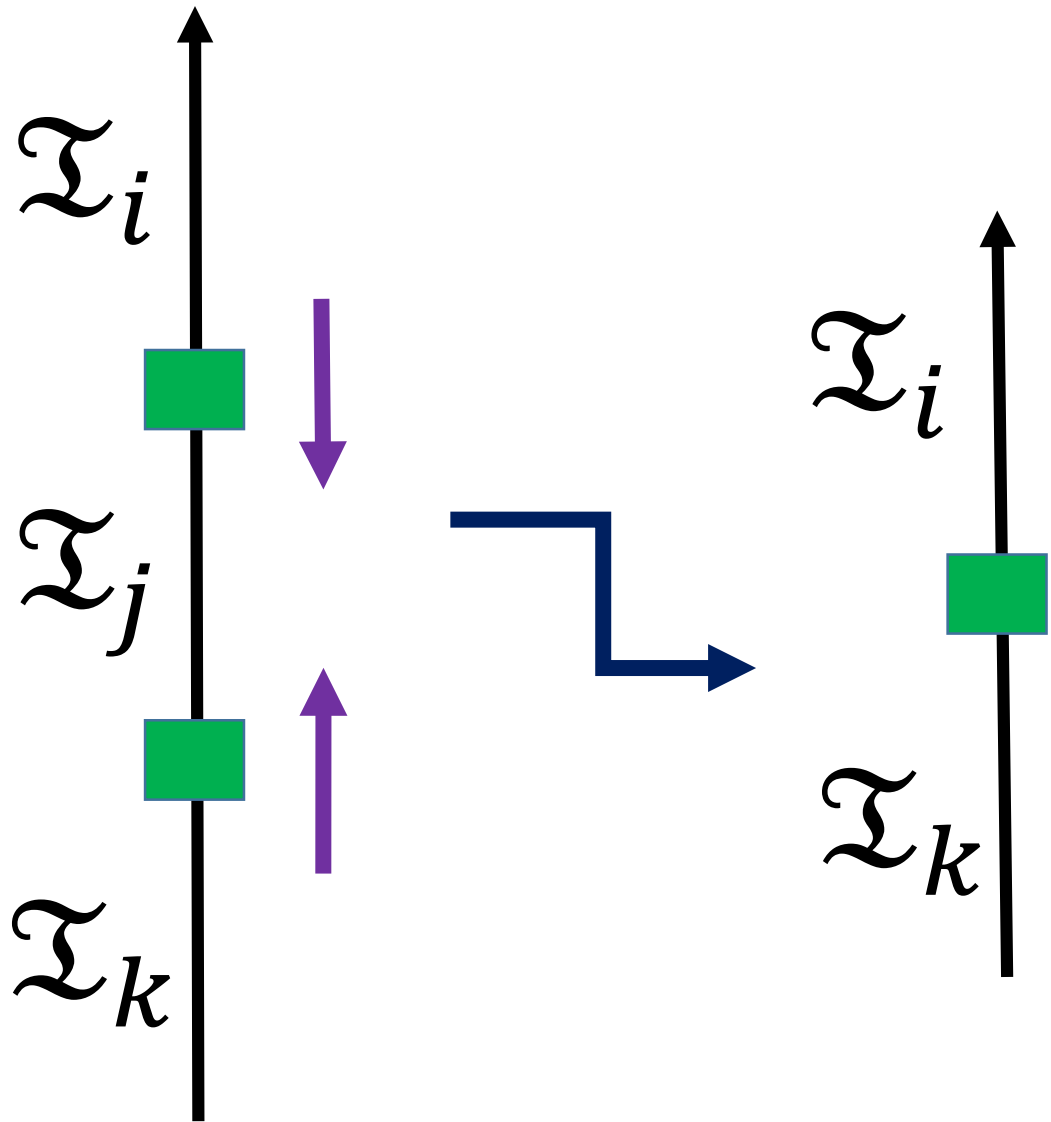
Choose a half-plane $\mathbb{H} \subset \mathbb{C}$, and a phase ζ

$\hat{R}_{ij} :=$ Vector space of local bc changing operators
between $\mathfrak{I}_j(\zeta)$ and $\mathfrak{I}_i(\zeta)$



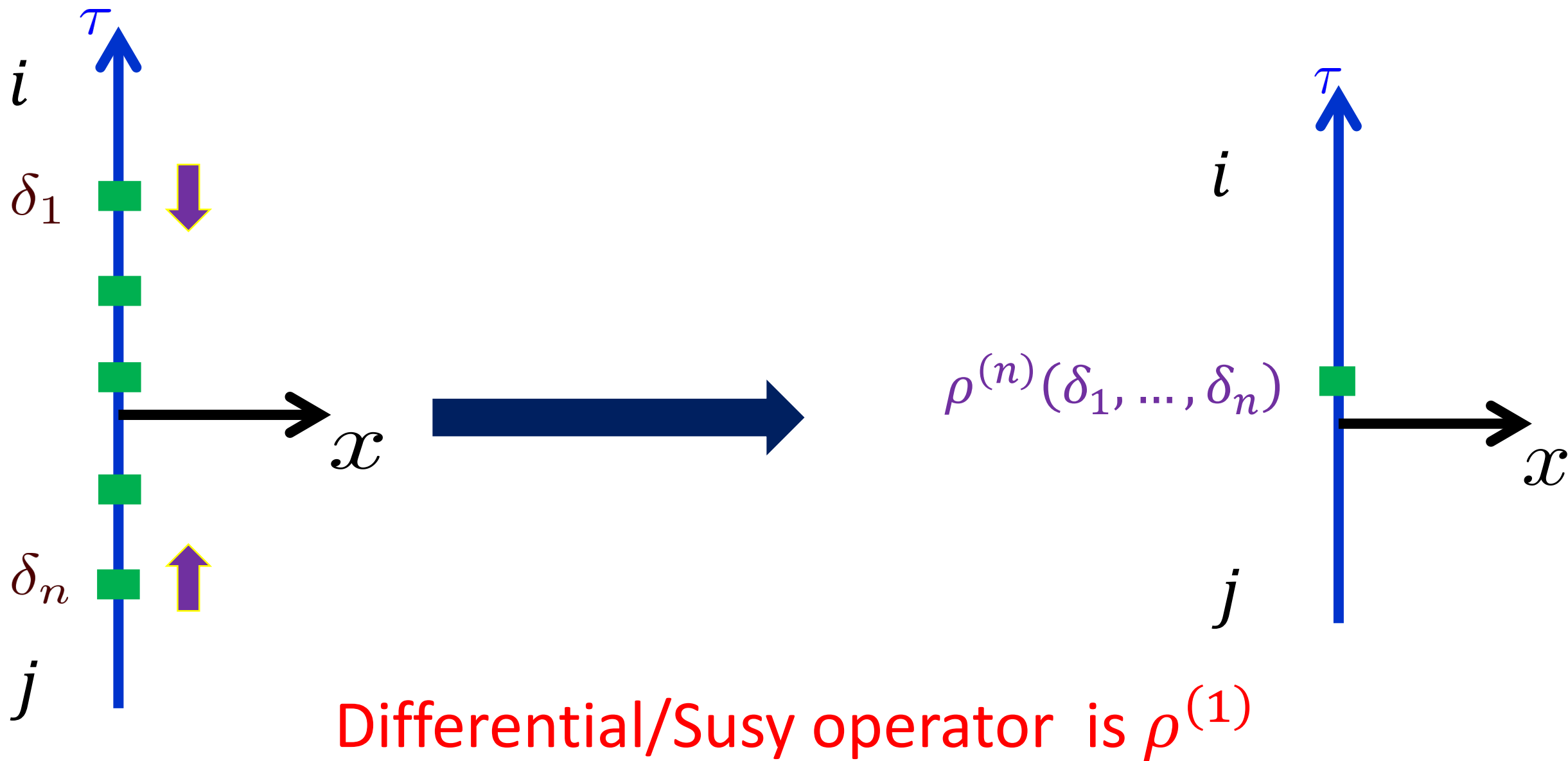
\hat{R}_{ij} is a chain
complex

$\hat{R} := \bigoplus_{ij} \hat{R}_{ij}$ Is An Algebra



$$\rho_{ijk}^{(2)}: \hat{R}_{ij} \otimes \hat{R}_{jk} \rightarrow \hat{R}_{ik}$$

\hat{R} has higher "OPE Products"



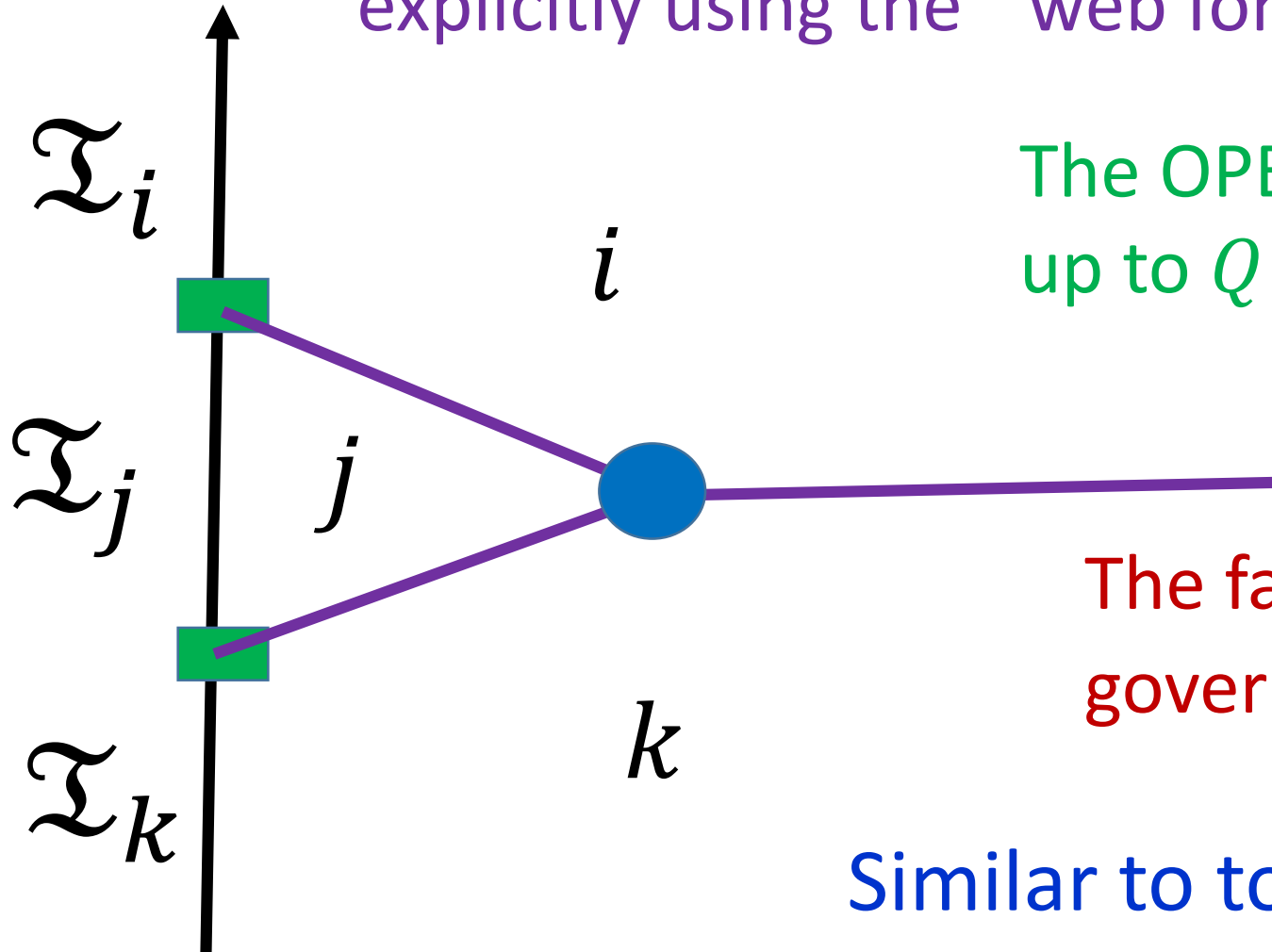
Web Formalism ($\alpha = \partial W$)

These multiplications can be constructed explicitly using the “web formalism” of GMW.

The OPE $\rho^{(2)}$ is only associative up to Q -exact terms.

The failure of associativity is governed by $\rho^{(3)}$

Similar to topological string theory.



Sources For The Web Formalism

Algebra of the Infrared: String Field Theoretic Structures in Massive $\mathcal{N} = (2, 2)$ Field Theory In Two Dimensions

Daive Gaiotto,¹ Gregory W. Moore,² and Edward Witten³

Algebra of the infrared and secondary polytopes

M. Kapranov, M. Kontsevich, Y. Soibelman

August 13, 2014

An Introduction to the Web Formalism

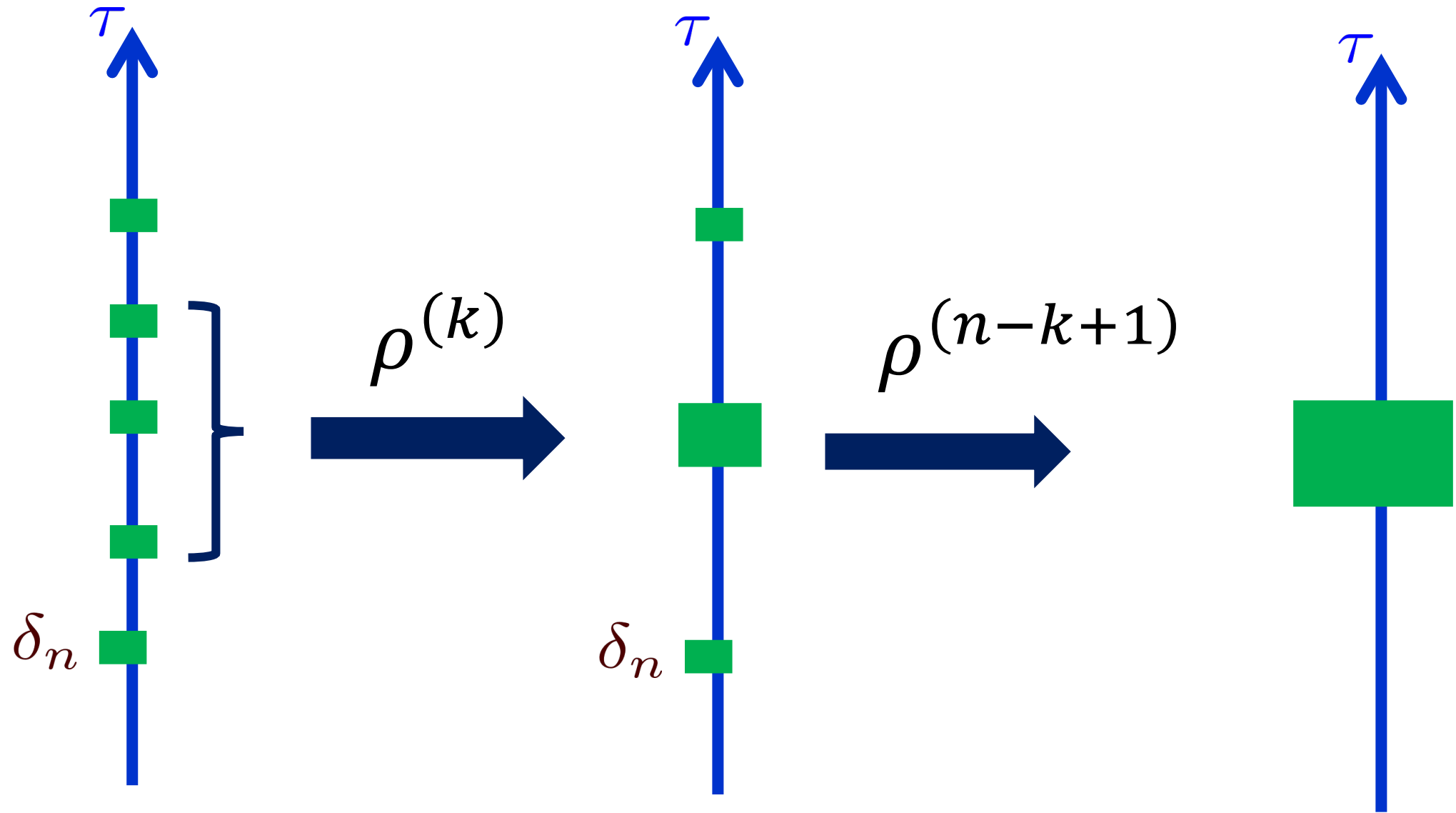
Daive Gaiotto,¹ Gregory W. Moore,² Edward Witten³



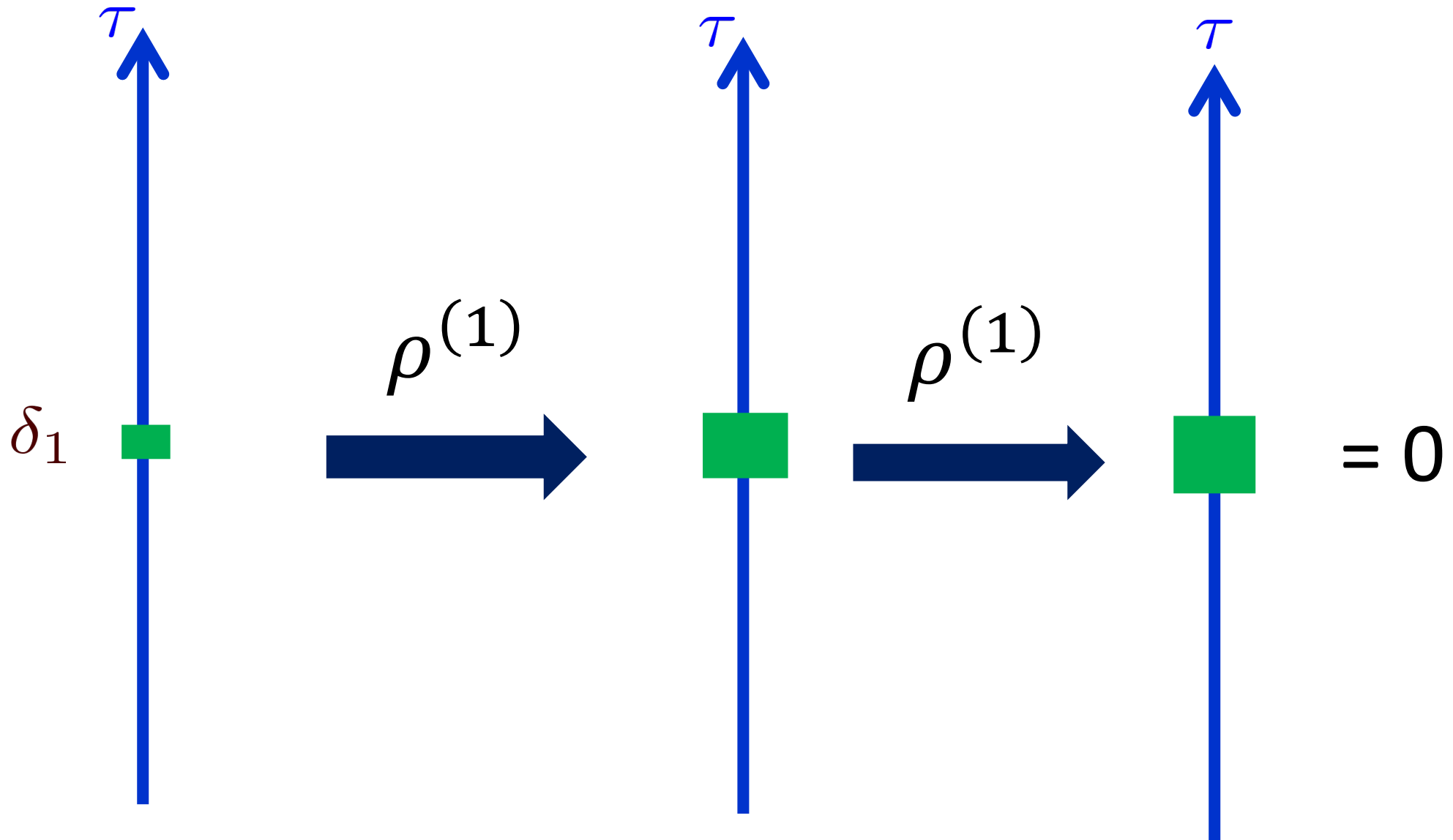
Only 51 pages !!!

\hat{R} As An A_∞ – Algebra

Sum with signs must vanish: Quadratic relations on the $\rho^{(n)}$: A_∞ – relations



An A-infty Algebra Is A Chain Complex



There is a notion of homotopy equivalence
of A_∞ -algebras.

It extends the notion of homotopy equivalence
of chain complexes, and says how the OPE's are
related to each other.

\hat{R} As A Category (When $\alpha = d W$)

Choose a half-plane $\mathbb{H} \subset \mathbb{C}$

Objects = thimbles \mathfrak{T}_i

$$\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j) = \begin{cases} \hat{R}_{ij} & Z_{ij} \in \mathbb{H} \\ \mathbb{Z} & i = j \\ 0 & Z_{ij} \notin \mathbb{H} \end{cases}$$

$$\hat{R}(X, g_{I\bar{J}}, W, \zeta)$$

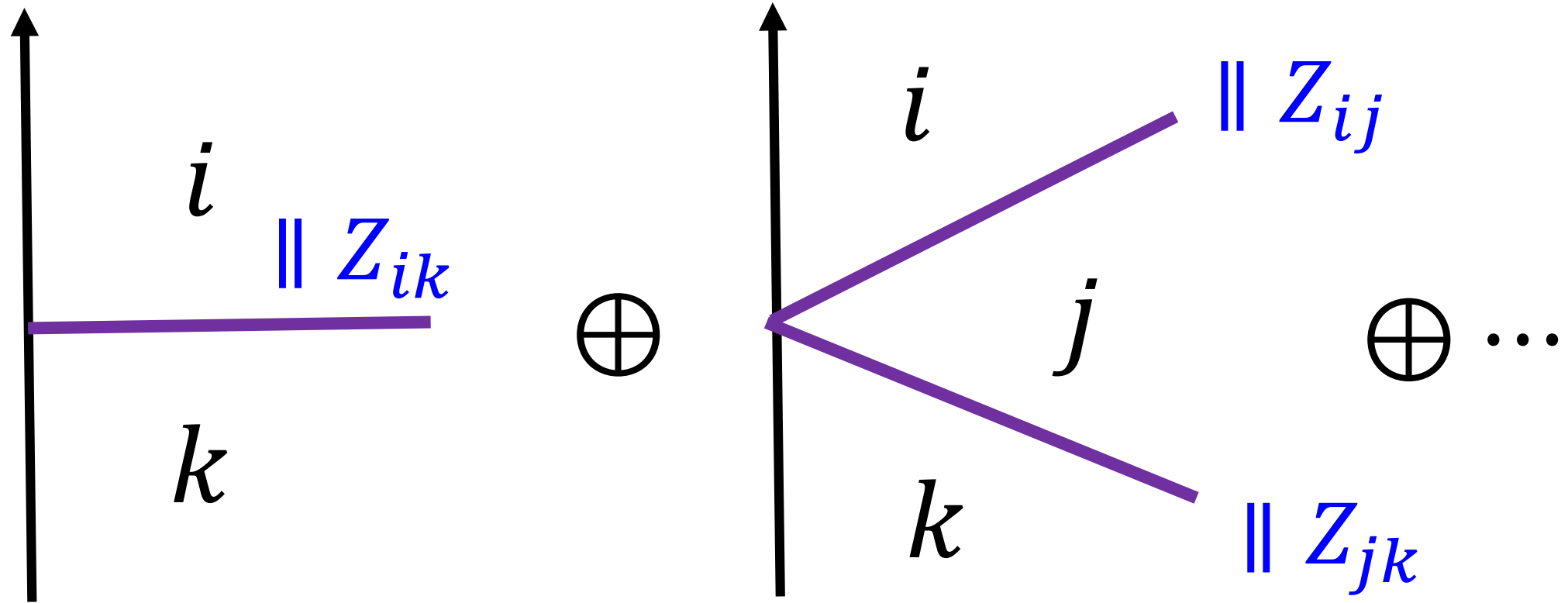
The Product Formula

\hat{R}_{ij} can be written in terms of R_{ij} [GMW]

$$\hat{R} := \bigoplus_{ij} \hat{R}_{ij} e_{ij} = \underbrace{\bigotimes_{z_{ij} \in \mathbb{H}} (\mathbb{Z} 1 + R_{ij} e_{ij})}_{\text{phase ordered!}} \underbrace{\quad}_{\text{upper triangular!}}$$

Summands correspond to central charges
 $Z_{i,i_1}, Z_{i_1,i_2}, \dots, Z_{i_n,k}$ whose phases are
clockwise ordered in a half plane

$$\hat{R}_{ik} = R_{ik} \oplus R_{ij} \otimes R_{jk} \oplus \dots$$



Naïve Differential On \hat{R}_{ik}

$$\hat{R}_{ik} = R_{ik} \oplus R_{ij} \otimes R_{jk} \oplus \dots$$

$$\hat{Q}^{naive} = Q_{ik} \oplus (Q_{ij} \otimes 1 + 1 \otimes Q_{jk}) \oplus \dots$$

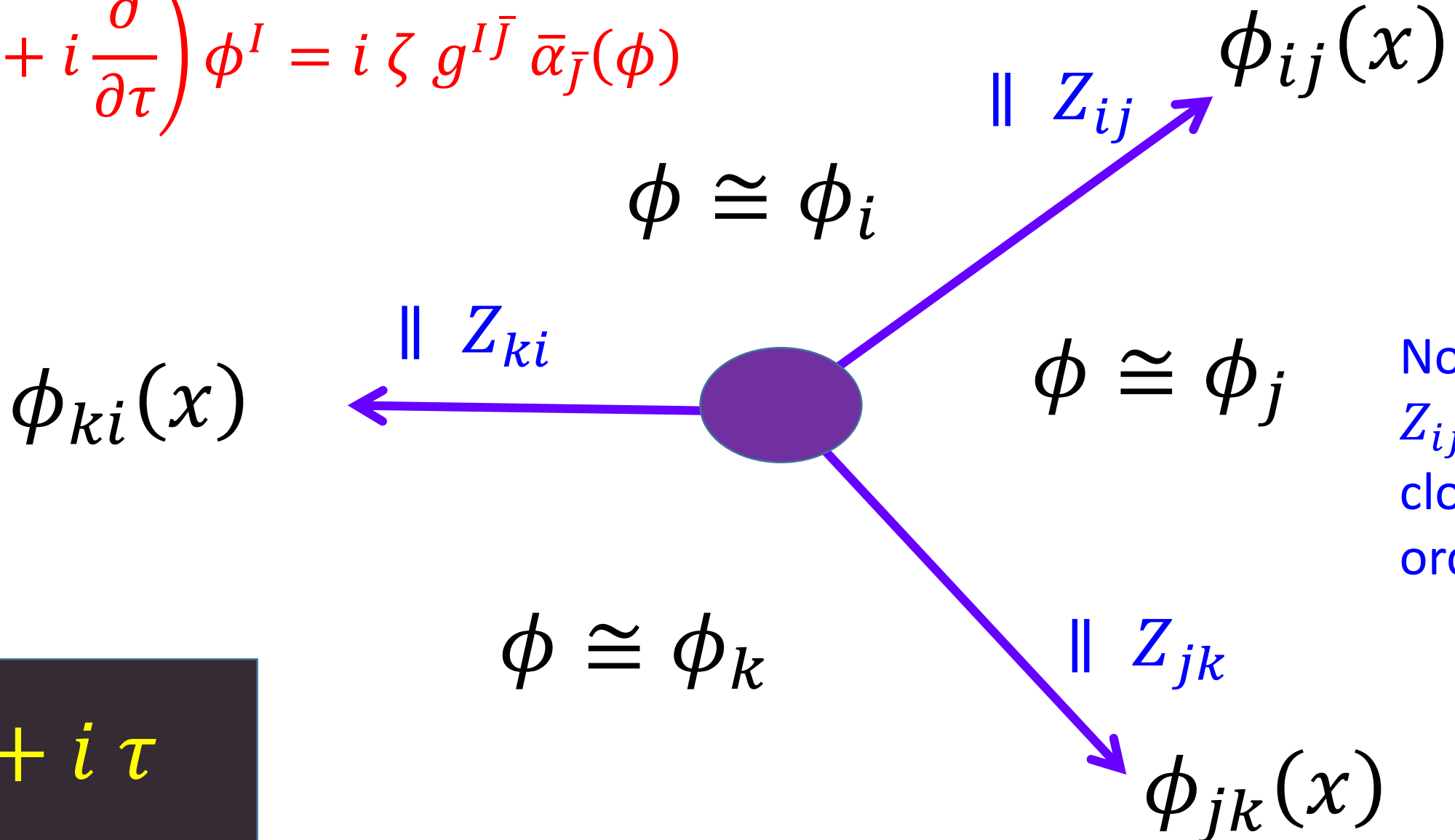
PHYSICALLY WRONG!

What went wrong?

We missed important instanton effects

Domain Wall Junctions: ζ –instanton equation

$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial \tau} \right) \phi^I = i \zeta g^{I\bar{J}} \bar{\alpha}_{\bar{J}}(\phi)$$



$x + i\tau$

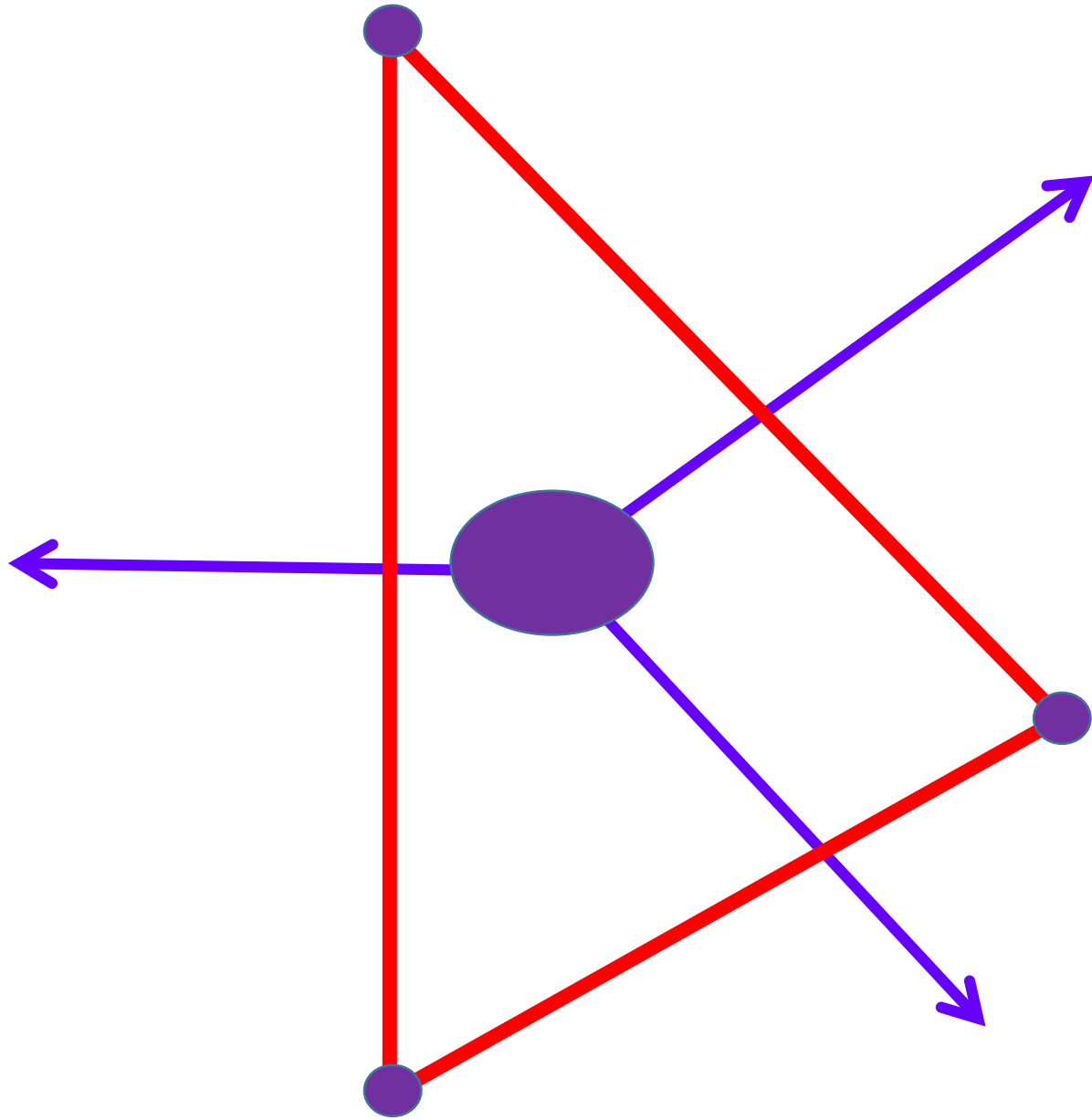


Image in
W- plane

Generalizes to
multi-valent
 n –vertices for $n > 3$

Explicit examples studied in

S. M. Carroll, S. Hellerman and M. Trodden, “Domain wall junctions are 1/4 - BPS states,” Phys. Rev. D 61, 065001 (2000) [hep-th/9905217].

G. W. Gibbons and P. K. Townsend, “A Bogomolny equation for intersecting domain walls,” Phys. Rev. Lett. 83, 1727 (1999) [hep-th/9905196].

H. Oda, K. Ito, M. Naganuma and N. Sakai, “An Exact solution of BPS domain wall junction,” Phys. Lett. B 471, 140 (1999) [hep-th/9910095].

Interior Amplitude

$$\phi_{ij} \otimes \phi_{jk} \otimes \phi_{ki} \in R_{ij} \otimes R_{jk} \otimes R_{ki}$$

Counting solutions defines an “interior amplitude”

$$\beta_{ijk} \in R_{ij} \otimes R_{jk} \otimes R_{ki}$$

Summing over all such cyclic fans defines an L_∞ –algebra

$$R_c = \bigoplus_{\text{cyclic fans}} R_{ij} \otimes \cdots \otimes R_{li}$$

β is a Maurer-Cartan element in an L_∞ algebra

Interior Amplitude Induces A Chain Map

$$\beta_{ijk} \in R_{ij} \otimes R_{jk} \otimes R_{ki}$$

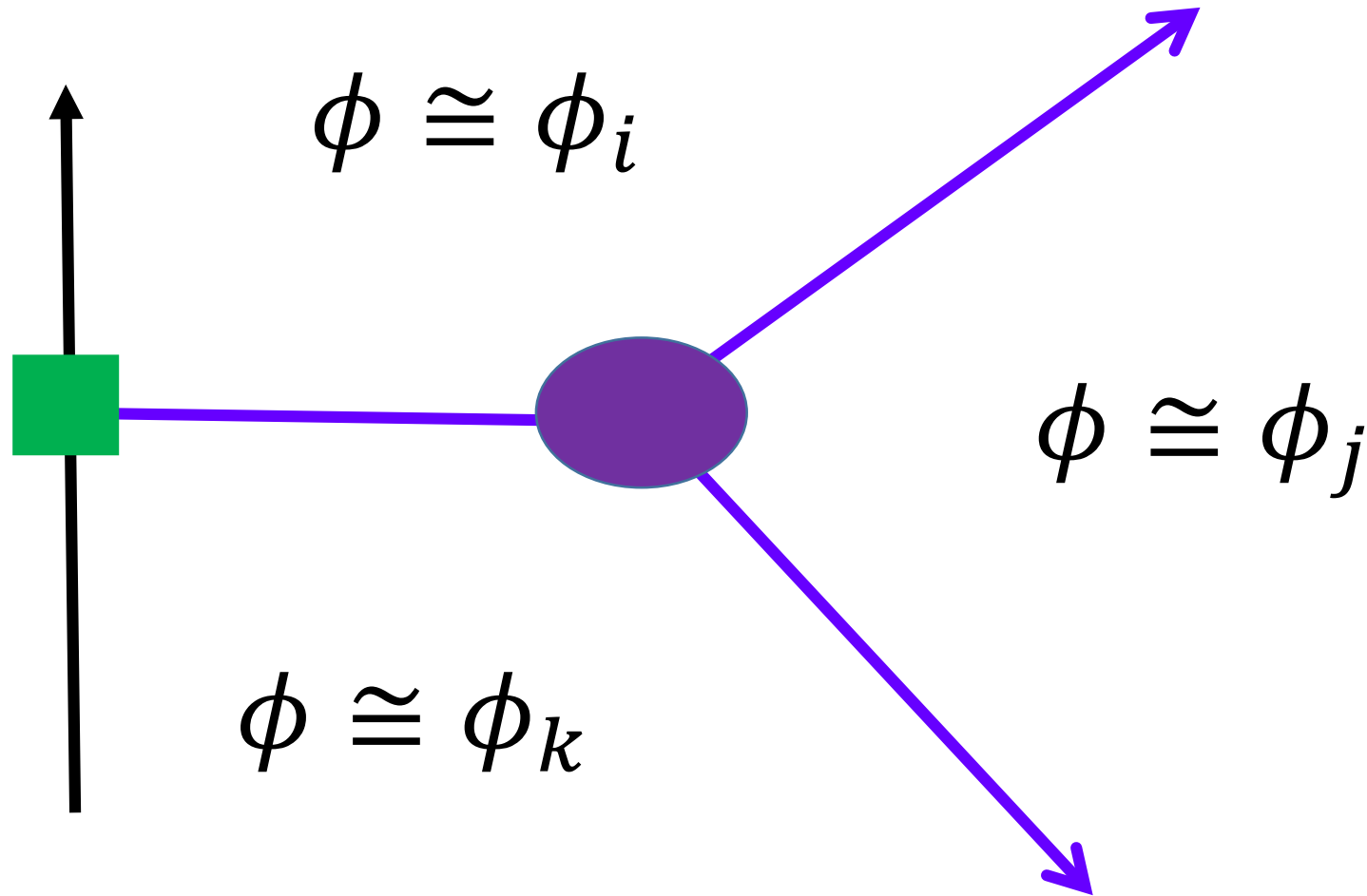
$$CPT \Rightarrow \exists \text{ (deg=-1) contraction: } K: R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}$$

Chain map:

$$M(\beta_{ijk}): R_{ik} \rightarrow R_{ij} \otimes R_{jk}$$

$M(\beta_{ijk})$ defines new component of differential on \hat{R}_{ik}

$$\hat{Q}_{ik}: R_{ik} \rightarrow R_{ij} \otimes R_{jk}$$



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Categorical Wall-Crossing

$$\text{IF: } (X, g_{I\bar{J}}, W, \zeta)_1 \sim (X, g_{I\bar{J}}, W, \zeta)_2$$

$$\text{THEN: } \hat{R}(X, g_{I\bar{J}}, W, \zeta)_1 \sim \hat{R}(X, g_{I\bar{J}}, W, \zeta)_2$$

\Rightarrow how the R_{ij} complexes change (up to h.e.)

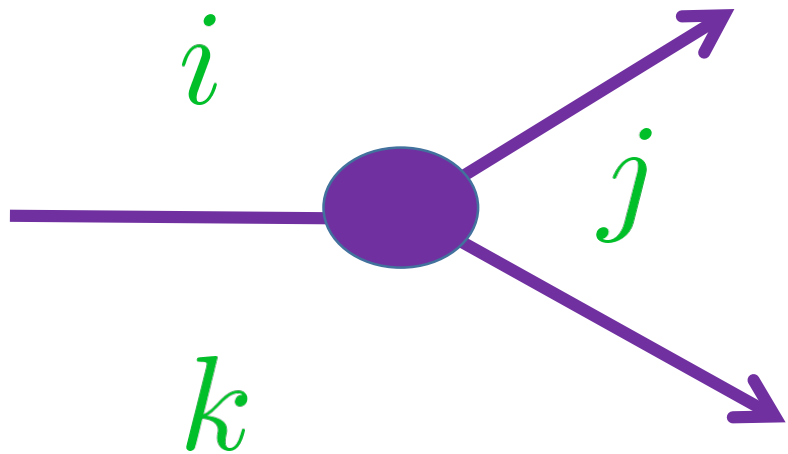
Definition: Cones In Homological Algebra

If $f: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a chain map

$$\mathcal{C}_1 := (V_1, F_1, Q_1) \quad \mathcal{C}_2 := (V_2, F_2, Q_2)$$

Then $\text{Cone}(f)$ is the new chain complex with

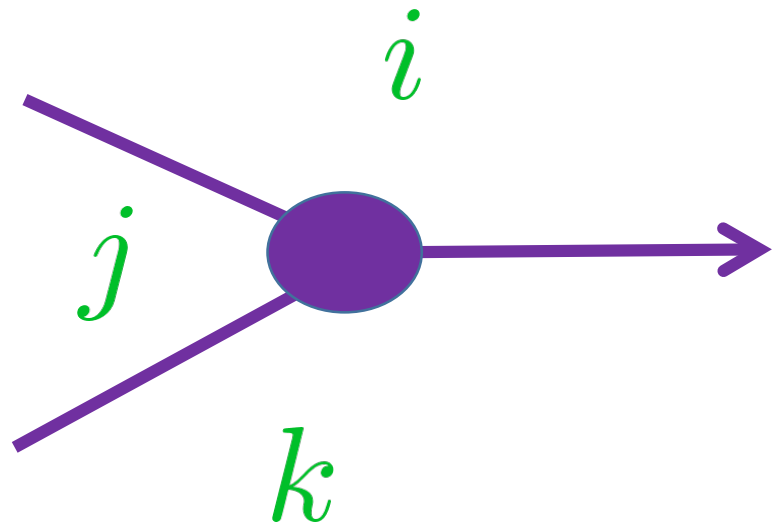
$$V := V_2 \oplus V_1[-1] \quad Q_{\text{Cone}(f)} = \begin{pmatrix} Q_2 & f \\ 0 & -Q_1 \end{pmatrix}$$



Defines a chain map

$$M^L(\beta_{ijk}): R_{ik}^L[1] \rightarrow R_{ij}^L \otimes R_{jk}^L$$

$Z_{ij} \parallel Z_{jk}$



Defines a chain map

$$M^R(\beta_{ikj}): R_{ij}^R \otimes R_{jk}^R \rightarrow R_{ik}^R$$

Solving Cat. Wall Crossing For $\alpha = \partial W$

An elegant way of solving the wall-crossing constraint

$$R_{ik}^R \sim \text{Cone}(M^L(\beta_{ijk}): R_{ik}^L[1] \rightarrow R_{ij}^L \otimes R_{jk}^L)$$

$$R_{ik}^L \sim \text{Cone}(M^R(\beta_{ikj}): R_{ij}^R \otimes R_{jk}^R \rightarrow R_{ik}^R)$$

Conversely, if $\hat{R}^L \sim \hat{R}^R$ then, up to homotopy,

$$R_{ik}^L \text{ and } R_{ik}^R$$

are related by cone constructions as above

$$R_{ik}^R \sim \text{Cone}(M^L(\beta_{ijk}): R_{ik}^L[1] \rightarrow R_{ij}^L \otimes R_{jk}^L)$$

$$R_{ik}^L \sim \text{Cone}(M^R(\beta_{ikj}): R_{ij}^R \otimes R_{jk}^R \rightarrow R_{ik}^R)$$

Taking Euler characters gives the Cecotti-Vafa wall-crossing formula:

$$\mu_{ik}^R = \mu_{ik}^L + \mu_{ij}^L \mu_{jk}^L$$

$$\mu_{ik}^L = \mu_{ik}^R - \mu_{ij}^R \mu_{jk}^R$$

Cecotti-Vafa Cones



How do they generalize to the case with twisted masses?

- 1 Supersymmetric Quantum Mechanics And Homological algebra
- 2 2D $N=(2,2)$ Landau-Ginzburg Models
- 3 Thimble Branes & Their Local Operators
- 4 Categorical Wall-Crossing
- 5 Generalization To Twisted Masses
- 6 Relation To 3d Indices

Generalization To Twisted Masses

Work in progress with Ahsan Khan

Definition: A *cyclic fan of charges* is a cyclically-ordered set

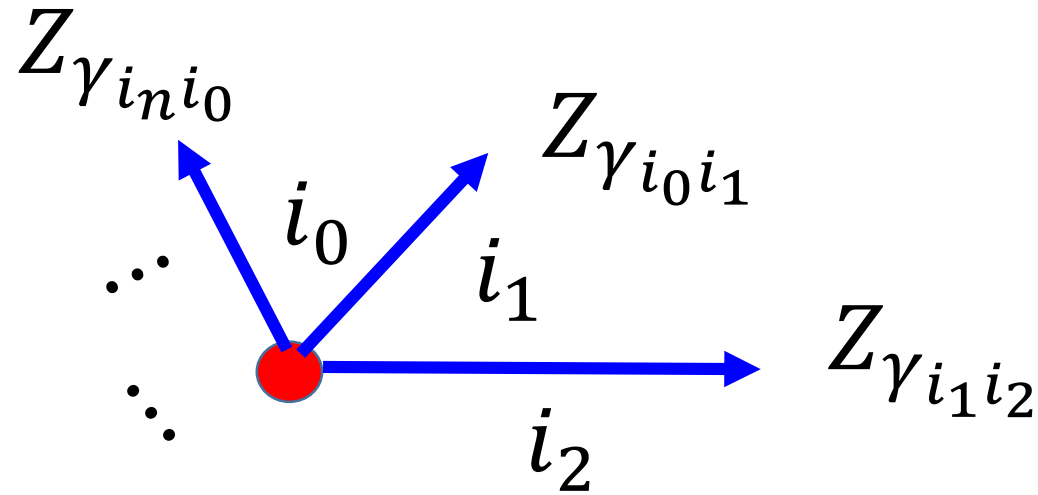
$$\{ \gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0} \}$$

So that the phases of $Z_{\gamma_{k,k+1}}$ are *monotonically*
decreasing (clockwise)

New ingredient: Successive $Z_{\gamma_k, \gamma_{k+1}}$ can be parallel.

Representations Of (Irreducible) Webs

Irreducible fans: $\{\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0}\}$ $i_k \neq i_{k+1}$

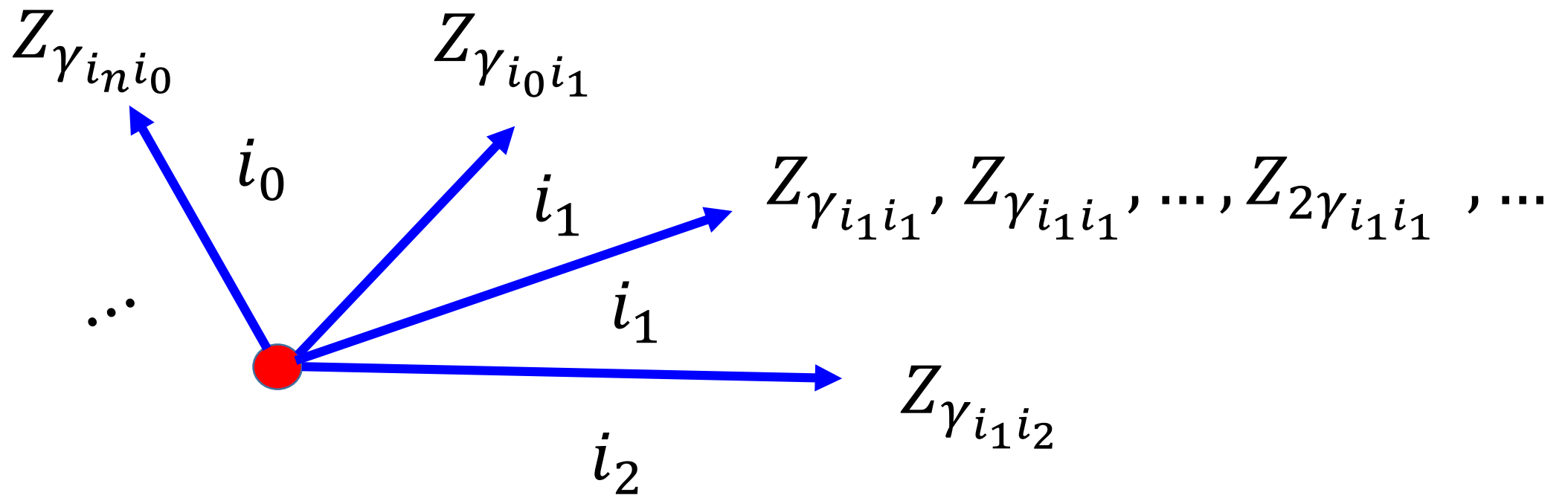


$$R_{\gamma_{i_0 i_1}} \otimes R_{\gamma_{i_1 i_2}} \otimes \dots \otimes R_{\gamma_{i_n i_0}}$$

GMW web formalism applies to give L_∞ algebra structure on the sum over all these pictures

Representations Of Generalized Webs

$$\{ \gamma_{i_0 i_1}, \gamma_{i_1 i_1}, \gamma_{i_1 i_1}, \dots, 2\gamma_{i_1 i_1}, \dots, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0} \}$$

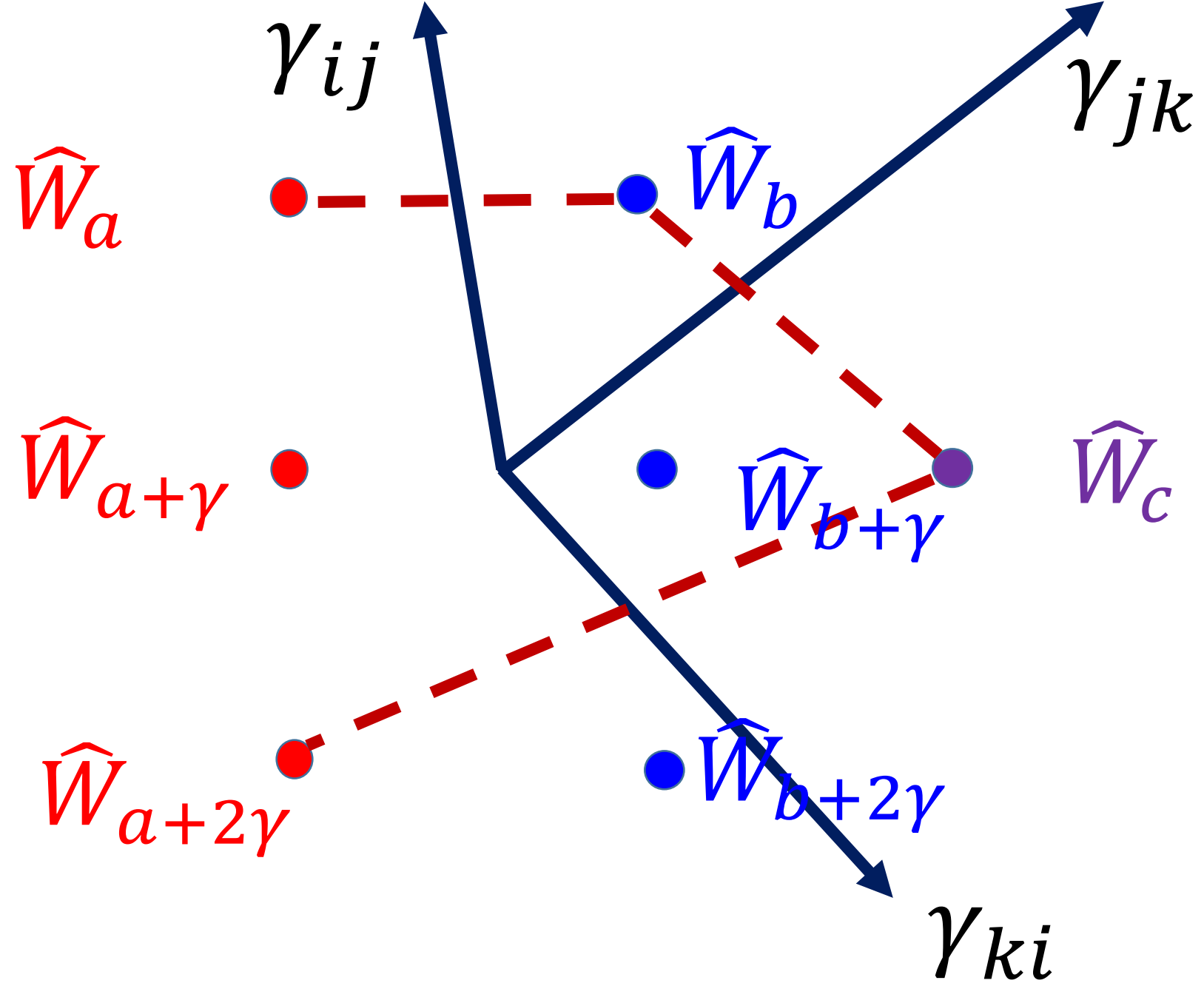


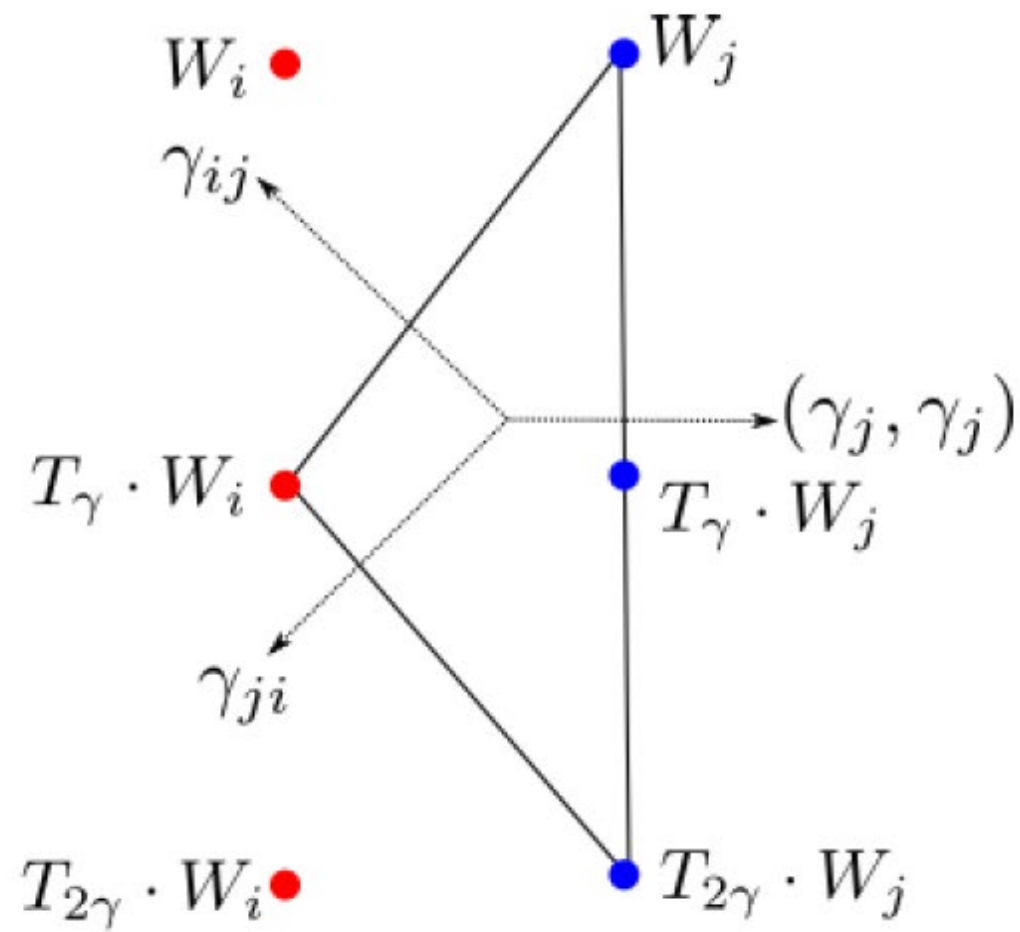
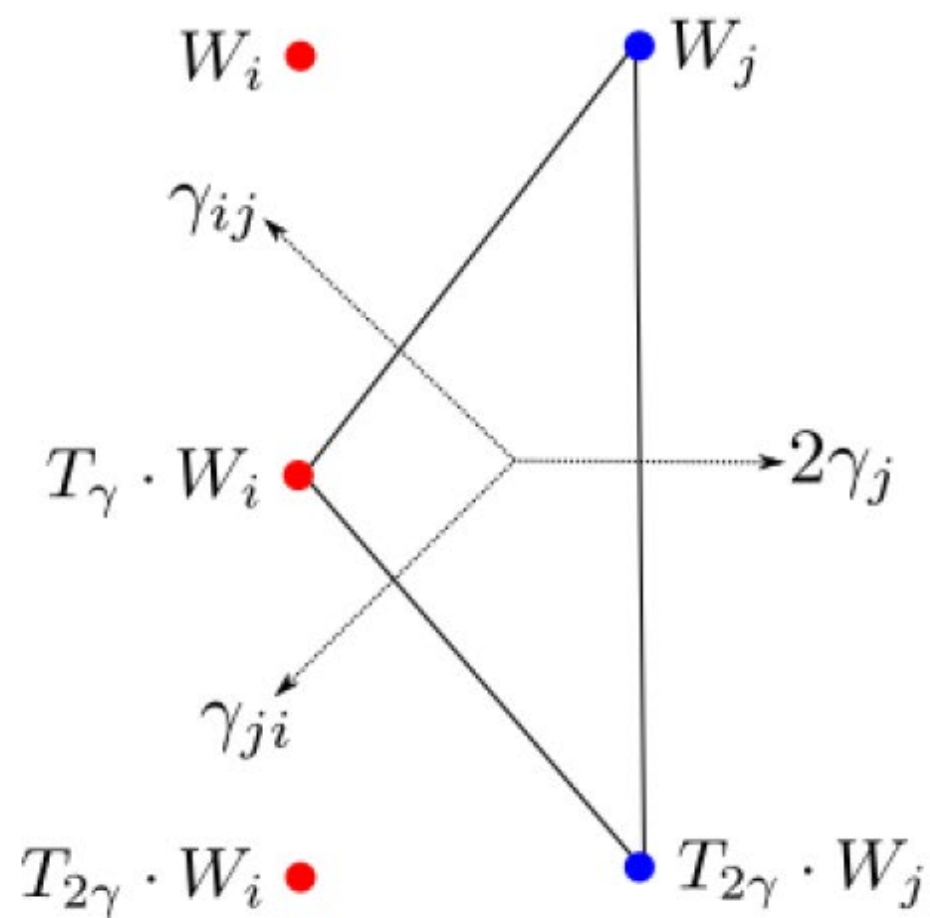
New Features Of Generalized Webs

$$\gamma_{i_0, i_1} + \gamma_{i_1, i_2} + \cdots + \gamma_{i_n, i_0} \neq 0$$

$$Z(\gamma_{i_0, i_1}) + Z(\gamma_{i_1, i_2}) + \cdots + Z(\gamma_{i_n, i_0}) \neq 0$$

Violates the “no-force condition.”



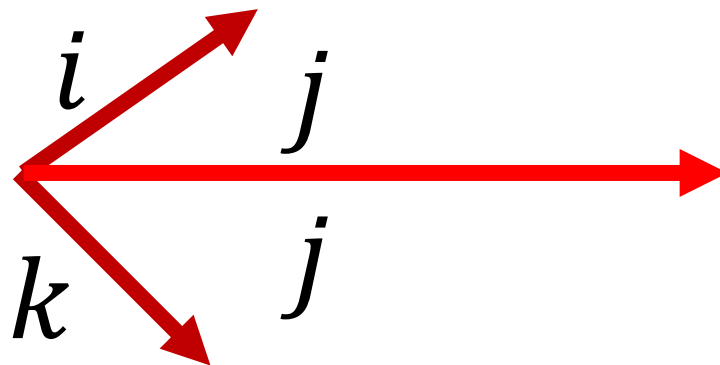


Representations Of Generalized Webs

$$\{ \gamma_{i_0 i_1}, \gamma_{i_1 i_1}, \gamma_{i_1 i_1}, \dots, 2\gamma_{i_1 i_1}, \dots, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0} \}$$

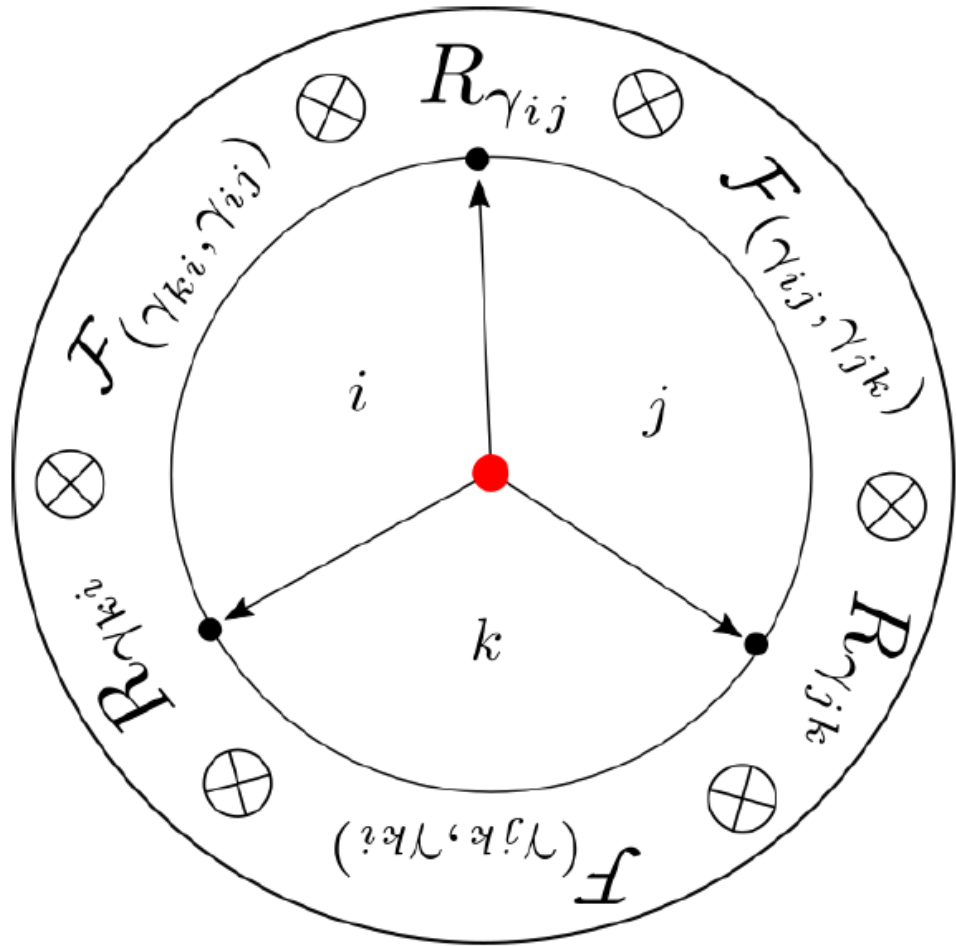
$\mathcal{F}_{\gamma_{jj}}$:= Graded Fock space on $R_{\gamma_{jj}}$

$$i \neq j \neq k \neq i \quad \mathcal{F}_{\gamma_{ij}, \gamma_{jk}} := \bigotimes_{\gamma_{ij} < \gamma_{jj} < \gamma_{jk}} \mathcal{F}_{\gamma_{jj}}$$



$$R_c = \bigoplus_{\{\gamma_{i_0 i_1}, \dots, \gamma_{i_n i_0}\} \text{ irred}}$$

$$\mathcal{F}_{\gamma_{i_n i_0}, \gamma_{i_0 i_1}} \otimes R_{\gamma_{i_0 i_1}} \otimes \dots \otimes \mathcal{F}_{\gamma_{i_{n-1} i_n}, \gamma_{i_n i_0}} \otimes R_{\gamma_{i_n i_0}}$$



Conjecture: Has a natural L_∞ structure associated with generalized webs.

Checked in several special cases.

A_∞ – Category Of Branes

$\hat{R}_{ij} :=$ Sum over all half-plane fans with Fock spaces inserted when the vacuum doesn't change.

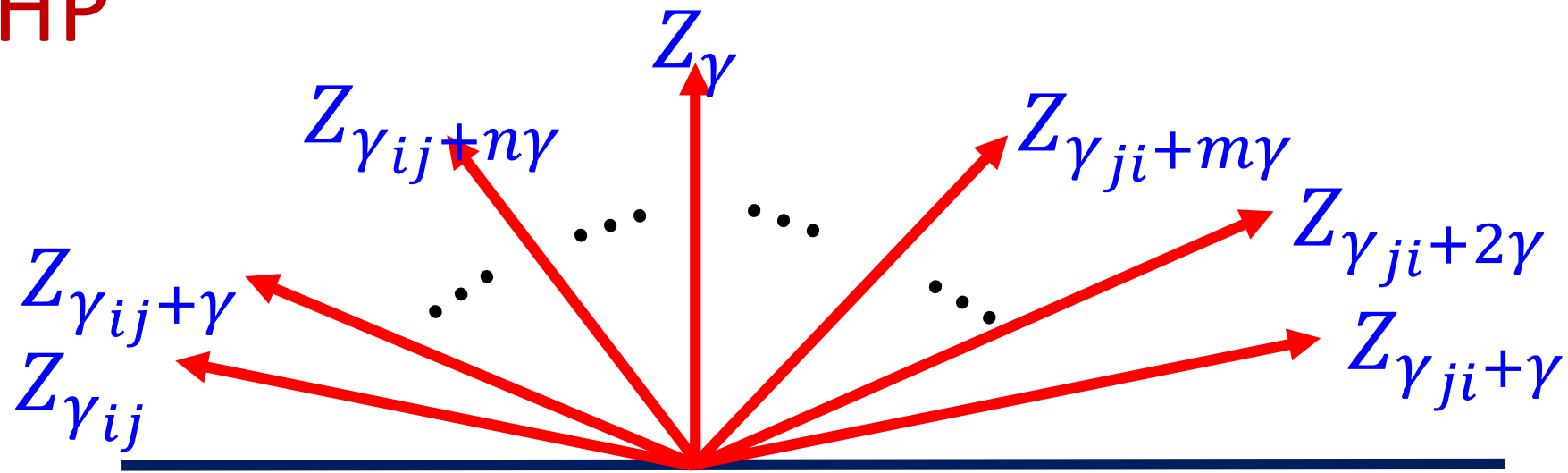
Conjecture: (\hat{R}, R_c) has the structure of an LA_∞ -category (= open closed homotopy category)

$$R_c = \text{Tr } \hat{R}^{opp} \hat{R}$$

Example: The Peacock Pattern

Special case: Two vacua ϕ_i, ϕ_j and rank $\Gamma = \mathbb{Z}\gamma$ of rank 1.

$\mathbb{H} = \text{UHP}$



$$\hat{R} = \bigotimes_{(n=0)}^{\infty} \begin{pmatrix} \mathbb{Z} & R_{\gamma_{ij}+n\gamma} \\ 0 & \mathbb{Z} \end{pmatrix} \bigotimes_{k \geq 1} \begin{pmatrix} \mathcal{F}[R_{k\gamma_{ii}}] & 0 \\ 0 & \mathcal{F}[R_{k\gamma_{jj}}] \end{pmatrix} \bigotimes_{n=\infty}^1 \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\gamma_{ji}+n\gamma} & \mathbb{Z} \end{pmatrix}$$

$$\hat{R} = \bigotimes_{(n=0)}^{\infty} \begin{pmatrix} \mathbb{Z} & R_{\gamma_{ij}+n\gamma} \\ 0 & \mathbb{Z} \end{pmatrix} \bigotimes_{k \geq 1} \begin{pmatrix} \mathcal{F}[R_{k\gamma_{ii}}] & 0 \\ 0 & \mathcal{F}[R_{k\gamma_{ii}}] \end{pmatrix} \bigotimes_{n=0}^{\infty} \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\gamma_{ji}+n\gamma} & \mathbb{Z} \end{pmatrix}$$

Conjecture: Matrix elements of \hat{R} are morphism spaces in an A_{∞} – category with multiplications induced from R_c

$$\hat{R}_{ii} = \mathcal{F}_{ii} \oplus \mathcal{A}_{\gamma_{ij}} \otimes \mathcal{F}_{jj} \otimes \mathcal{A}_{\gamma_{ji}}$$

$$\hat{R}_{ji} = \mathcal{F}_{jj} \otimes \mathcal{A}_{\gamma_{ji}}$$

$$\hat{R}_{ij} = \mathcal{A}_{\gamma_{ij}} \otimes \mathcal{F}_{jj}$$

$$\hat{R}_{jj} = \mathcal{F}_{jj}$$

$$\mathcal{F}_{ii} := \bigotimes_{k \geq 1} \mathcal{F}[R_{k\gamma_{ii}}]$$

$$\mathcal{A}_{\gamma_{ij}} := \bigoplus_{k \geq 0} R_{\gamma_{ij}+k\gamma}$$

$$\mathcal{A}_{\gamma_{ji}} := \bigoplus_{k \geq 1} R_{\gamma_{ji}+k\gamma}$$

$$\mathcal{F}_{jj} := \bigotimes_{k \geq 1} \mathcal{F}[R_{k\gamma_{jj}}]$$

Generalization Of The Cone Formula

$$\hat{R} = \bigotimes_{(n=0)}^{\infty} \begin{pmatrix} \mathbb{Z} & R_{\gamma_{ij}+n\gamma} \\ 0 & \mathbb{Z} \end{pmatrix} \bigotimes_{k \geq 1} \begin{pmatrix} \mathcal{F}[R_{k\gamma_{ii}}] & 0 \\ 0 & \mathcal{F}[R_{k\gamma_{ii}}] \end{pmatrix} \bigotimes_{n=0}^{\infty} \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\gamma_{ji}+n\gamma} & \mathbb{Z} \end{pmatrix}$$

$$\hat{R}_{ii} = \mathcal{F}_{ii} \oplus \mathcal{A}_{\gamma_{ij}} \otimes \mathcal{F}_{jj} \otimes \mathcal{A}_{\gamma_{ji}}$$

After wall-crossing through $Im \left(Z_{\gamma_{ij}} \bar{Z}_{\gamma} \right)$ we write \hat{R}' in the reverse order...

$$\hat{R}'_{ii} = \mathcal{F}'_{ii} \quad \hat{R}'_{ij} = \mathcal{F}'_{ii} \otimes \mathcal{A}'_{\gamma_{ij}}$$

$$\hat{R}'_{ji} = \mathcal{A}'_{\gamma_{ji}} \otimes \mathcal{F}'_{ii} \quad \hat{R}'_{jj} = \mathcal{F}'_{jj} \oplus \mathcal{A}'_{\gamma_{ji}} \otimes \mathcal{F}'_{ii} \otimes \mathcal{A}'_{\gamma_{ij}}$$

Soluton Of Categorized Wall-Crossing: 2 Vacua

Demading homotopy equivalence of the A_∞ –categories determined by \hat{R} and \hat{R}'

$$\mathcal{F}'_{\gamma_{ii}} \cong \hat{R}_{ii}$$

$$F'_i = F_i + A_{ij}F_jA_{ji}$$

$$\mathcal{A}'_{\gamma_{ij}} \cong \hat{R}_{ii}^! \otimes \mathcal{A}_{\gamma_{ij}} \otimes \mathcal{F}_{\gamma_{jj}}$$

$$A'_{ij} = (F_i + A_{ij}F_jA_{ji})^{-1} A_{ij}F_j$$

$$\mathcal{A}'_{\gamma_{ji}} \cong \mathcal{F}_{\gamma_{jj}} \otimes \mathcal{A}_{\gamma_{ji}} \otimes \hat{R}_{ii}^!$$

$$A'_{ji} = (F_i + A_{ij}F_jA_{ji})^{-1} F_jA_{ji}$$

$$\mathcal{F}'_{\gamma_{jj}} \cong \hat{R}_{ii}^! \otimes \mathcal{F}_{\gamma_{ii}} \otimes \mathcal{F}_{\gamma_{jj}}$$

$$F'_j = (F_j + A_{ij}F_jA_{ji})^{-1} F_iF_j$$

Here $\hat{R}_{ii}^!$ means the “Koszul co-dual” of \hat{R}_{ii}

Koszul duality has played an important role in
several recent works

Costello 2017;

Dimofte (String-Math 2017) ;

Gaiotto & Oh; Costello & Paquette,

1

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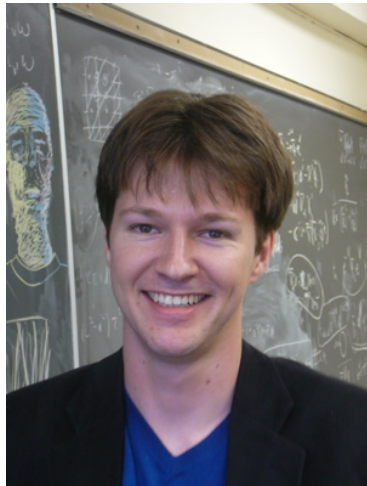
6

Relation To Three-Dimensional SQFT

RELATION TO 3D INDEX



With nontrivial input from Andy and Tudor



Motivation

Recent striking conjecture by Garoufalidis, Gu, and Marino,
“Peacock Patterns And Resurgence In Complex Chern-Simons Theory”

They observed a relation between the 3d index

$$I_T(q, x) = \text{Tr}_{\mathcal{H}_T} (-1)^F q^{\frac{1}{2}R + j_3} x^e$$

T : 3d SUSY class R theory associated
with a hyperbolic knot complement $T(M_K)$

and Stokes matrices related to
thimbles in complex Chern-Simons theory on M_K

We give it a natural context and state a conjecture about PDE's (Kapustin-Witten equations) which implies the GGM conjecture.

In fact, the conjecture has been stated before by Victor Mikhaylov (2017) for different reasons.

$T(M_3)$

$T(M_3)$ is a theory obtained by reduction of 6d (2,0) on M_3
with topological twist (class R)

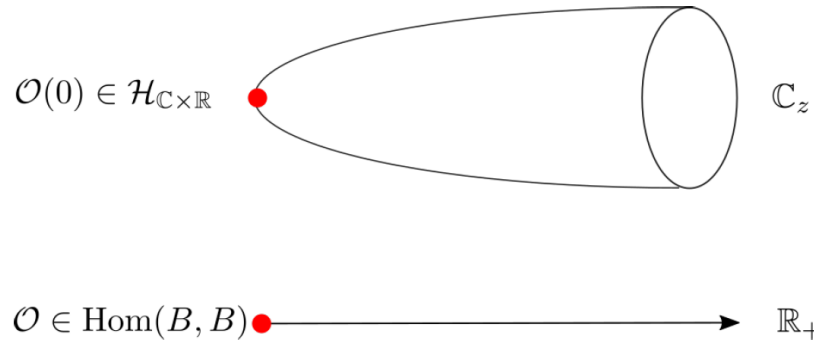
[Dimofte, Gaiotto, Gukov; Terashima-Yamazaki]

Consider $T(M_3)$ on $\mathcal{C} \times \mathbb{R}_t$



With “holomorphic-topological twist” [Witten; Oh-Yagi]
we can identify the index $I_T(q, x)$ with the trace over the
Q-cohomology of local operators at the tip of the cigar.

KK Reduction on Cigar



Get a LG theory on half space with boundary condition \mathfrak{B}_{cigar}

Q –closed local operators on $\mathcal{C} \times \mathbb{R}_t$ \longleftrightarrow Boundary operators for \mathfrak{B}_{cigar}

$$I_{T(M_3)}(q, x) = \text{Tr}_{\text{Hop}(\mathfrak{B}_{cigar}, \mathfrak{B}_{cigar})} (-1)^F q^{J_1} x^{J_2}$$

$$\text{LG Model} = \text{CSLG}[M_3]$$

$$6d (2,0) / M_3 \times \mathcal{C} \times \mathbb{R}_t$$

M_3




$$T[M_3] / \mathcal{C} \times \mathbb{R}_t$$

$U(1)_c$



$$5D \text{ SYM} / M_3 \times \mathbb{R}_+ \times \mathbb{R}_t$$

$U(1)_c$



$$LG / \mathbb{R}_+ \times \mathbb{R}_t$$

M_3



$$\text{CSLG}[M_3] / \mathbb{R}_+ \times \mathbb{R}_t$$

\cong

\mathcal{B}_{cigar}

\mathcal{B}_{Nahm}

Relation to KW Equations With Nahm bc's.

$$6d (2,0) / M_3 \times \mathcal{C} \times \mathbb{R}_t \xrightarrow{U(1)_c} 5d \text{ SYM} / M_3 \times \mathbb{R}_+ \times \mathbb{R}_t$$
$$ds^2 = g_{IJ}(x) dx^I dx^J + dy^2 + dt^2$$

Witten: BPS equations = KW equations on $M_3 \times \mathbb{R}_+$

$$y \rightarrow 0 \quad \phi^a = \frac{e^a}{y} + \mathcal{O}(1) \quad A^a = \omega^a + \mathcal{O}(y)$$

e^a, ω^a : Dreibein and spin connection
for Riemannian metric on M_3

Relation To $CSLG(M_3)$

KW equations with $\phi_y = 0$ and $A_y = 0$ are the ζ –soliton equations for $CSLG[M_3]$

Vacua of $CSLG[M_3]$: Flat connections σ_i

$$\mathcal{B}_{Nahm} \cong \sum_i \mathcal{E}_{\sigma_i} \mathcal{I}_{\sigma_i}$$

Chan-Paton complex $\mathcal{E}_{\sigma_i}(\mathcal{B}_{Nahm})$ is MSW for KW & Nahm bc's & $\mathcal{A} \rightarrow \sigma_i$ for $y \rightarrow \infty$

$$\mathcal{O} = (-1)^F q^{J_1} x^{J_2}$$

$$\begin{aligned}
 & \text{Tr}_{\text{Hop}}(\mathfrak{B}_{\text{cigar}}, \mathfrak{B}_{\text{cigar}}) \mathcal{O} \\
 = & \sum_i \text{Tr}_{\varepsilon_{\sigma_i}} \mathcal{O} \times \text{Tr}_{\text{Hop}}(\mathfrak{I}_{\sigma_i}, \mathfrak{I}_{\sigma_j}) \mathcal{O} \times \text{Tr}_{\varepsilon_{\sigma_j}} \mathcal{O}
 \end{aligned}$$

$$\text{Tr}_{\text{Hop}}(\mathfrak{I}_{\sigma_i}, \mathfrak{I}_{\sigma_j}) \mathcal{O} = S_{\sigma_i, \sigma_j}(q, x)$$

Stokes matrices for
Chern-Simons thimbles

Specialize to Hyperbolic Knot Complement

$M_3 = M_K$ knot complement in S^3 of a hyperbolic knot.

Among flat $SL(2, \mathbb{C})$ connections on M_K there is a distinguished one: σ_1

σ_1 : corresponds to the complete hyperbolic metric. $\mathcal{A} = \omega + i e$

Conjecture: $\mathcal{E}_{\sigma_i}(\mathcal{B}_{Nahm}) \cong \begin{cases} \mathbb{Z} & \sigma_i = \sigma_1 \\ 0 & \text{Else} \end{cases}$

Evidence

Independent evidence for the conjecture given by Victor Mikhaylov for $M_3 \times \mathbb{R}_+$, M_3 complete hyperbolic

Gives a natural explanation of the remarkable observation of Garoufalidis-Gu-Marino

Extends uniqueness result of Mazzeo-Witten
from $\mathbb{R}^3 \times \mathbb{R}_+$ to $M_K \times \mathbb{R}_+$

$$A = \omega \quad \& \quad \phi = \coth y e$$

P. Kronheimer; S. He; V. Mikhaylov;

Interesting Generalization

L : Colored (by reps) link in M_3

Witten: Modify Nahm boundary condition
in 5d SYM with 't Hooft line L

Corresponds to a brane $\mathfrak{B}(L)$ in $\text{CSLG}[M_3]$

Up to homotopy equivalence,
it only depends on isotopy class of $L \subset M_3$

Potentially New Knot Invariants

Conjecture:

- a.) $\{\mathfrak{B}(L)\}$ with L running over isotopy classes of colored links in M_3 is a (full?) sub- A_∞ –category of $\mathfrak{Br}(LG(M_3))$
- b.) This category is a 3-manifold invariant
- c.) A_∞ - algebras $Hop(\mathfrak{B}(L), \mathfrak{B}(L))$ are (new?) colored link invariants.

Conclusion

Using the framework of GMW we derived a categorified version of the Cecotti-Vafa WCF

The framework can be extended to the case with twisted masses. Here there are qualitatively new features involving Fock spaces of periodic solitons and Koszul duality of complexes.

This circle of ideas applied to Chern-Simons-Landau-Ginzburg theory associated to M_3 gives a natural framework for interpreting a recent striking conjecture of GGM, and moreover suggests potentially new colored link invariants.