Orientifolds, Twisted Cohomology, and Self-Duality

A talk for I.M. Singer on his 85th birthday

MIT, May 23, 2009

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Work *in progress* with
Jacques Distler & Dan Freed

Outline

1. Motivation & Two Main Themes
2. What is an orientifold?
3. Worldsheet action: bosonic & super
4. The B-field twists the RR-field
5. The RR-field is self-dual: Twisted spin structure
6. How to sum over worldsheet spin structures
7. O-plane charge
8. A prediction
9. Précis

Motivation

- This talk is a progress report on work done over a period of several years with J. Distler and D. Freed

- I want to explain how an important subject in string theory-the theory of orientifolds makes numerous contact with the interests of Is Singer.

- Historically, orientifolds played an important role in the discovery of D-branes. They are also important because the evidence for the alleged ``landscape of string vacua'' (d=4, N=1, with moduli fixed) relies heavily on orientifold constructions.

- So we should put them on a solid mathematical foundation!

(even for type I the worldsheet theory has not been written)

Theme 1

Our first theme is that finding such a foundation turns out to be a nontrivial application of many aspects of modern geometry and topology:

Index theory

Geometry of anomaly cancellation

Twisted K-theory

Differential generalized cohomology

Quadratic functors, and the theory of self-dual fields

Is Singer’s work is closely related to all the above

Theme 2

Our second theme is the remarkable interplay between the worldsheet and spacetime formulations of the theory.

Recall that a basic ingredient in string theory is the space of maps:

$$\varphi : \Sigma \to X$$

Σ: 2d Riemannian surface

X: Spacetime endowed with geometrical structures: Riemannian,…

2d sigma model action:

$$\exp[-\int_{\Sigma} \frac{1}{2} \parallel d\varphi \parallel^2 + \cdots]$$

Based on this D. Friedan showed – while
Is Singer's student – that:

2D Conformal Field Theories on Σ	$\Longleftrightarrow$	Einstein metrics on X

It's a good example of a deep relation between worldsheet and spacetime structures.

Orientifolds provide an interesting example where topological structures in the world-sheet (short-distance) theory are tightly connected with structures in the space-time (long-distance) theory.

I will emphasize just one aspect of this:

We will see that a ``*twisted spin structure*'' on X is an essential ingredient both in worldsheet anomaly cancellation and in the formulation of the self-dual RR field on X.

What is an orientifold?

Let's warm up with the idea of a string theory ***orbifold***

$$\varphi : \Sigma \to Y$$

$\text{Smooth } Y \text{ has finite isometry group } \Gamma$

$\text{Gauge the } \Gamma\text{-symmetry:}$

Principal Γ bundle

$$\begin{tikzcd} \tilde{\Sigma} \arrow[r, "\tilde{\varphi}"] \arrow[d] & Y \arrow[d] \\ \Sigma \arrow[r, "\varphi"] & Y/\!\!/\Gamma \end{tikzcd}$$

Physical worldsheet

Spacetime groupoid

For ***orientifolds***, $\tilde{\Sigma}$ is orient<u>ed</u>,

In addition: $1 \to \Gamma_0 \to \Gamma \overset{\omega}{\to} \mathbb{Z}_2 \to 1$

$$\Gamma_0\colon \ \omega(\gamma) = +1 \qquad \Gamma_1\colon \ \omega(\gamma) = -1$$

On $\tilde{\Sigma}$: Γ_0: Orientation preserving
Γ_1: Orientation reversing

$$\begin{array}{ccc} \tilde{\Sigma} & \overset{\tilde{\varphi}}{\to} & Y \\ \downarrow & & \downarrow \\ \hat{\Sigma} & \overset{\hat{\varphi}}{\to} & Y /\!/ \Gamma_0 \\ \downarrow & & \downarrow \\ \Sigma & \overset{\varphi}{\to} & Y /\!/ \Gamma \end{array}$$

Orientation double cover → $\hat{\Sigma}$

<u>Unoriented</u> → Σ

$Y /\!/ \Gamma$ ← Space time

More generally: Spacetime X is an ``orbifold''
(Satake, Thurston…) with double cover X_w

$$
w_1(\Sigma) \in H^1(\Sigma, \mathbb{Z}_2) \qquad
\begin{array}{ccc}
\hat{\Sigma} & \stackrel{\hat{\varphi}}{\rightarrow} & X_w \\
\hat{\pi} \downarrow & & \downarrow \\
\Sigma & \stackrel{\varphi}{\rightarrow} & X
\end{array}
\qquad w \in H^1(X, \mathbb{Z}_2)
$$

There is an isomorphism:

$$\varphi^*(w) \cong w_1(\Sigma)$$

$\hat{\Sigma}$: Orientation double cover of Σ

For those – like me – who are afraid of stacks,

it is fine to think about the global quotient:

$$X_w = Y /\!\!/ \Gamma_0 \qquad\qquad X = Y /\!\!/ \Gamma$$

Just bear in mind that cohomology in this case really means Borel equivariant cohomology

$$H^j(Y/\!\!/\Gamma) := H^j_\Gamma(Y)$$

and we will again need to be careful about K

Worldsheet Measure

In string theory we integrate over ``worldsheets''

For the bosonic string, space of ``worldsheets'' is

$$S = \{(\Sigma, \varphi)\} = \mathrm{Moduli}(\Sigma) \times \mathrm{MAP}(\Sigma \to X)$$

$$\exp[-\int_{\Sigma/S} \frac{1}{2} \parallel d\varphi \parallel^2] \cdot \mathcal{A}_B$$

$$\mathcal{A}_B = \exp[2\pi i \int_{\Sigma/S} \varphi^*(B)]$$

B is locally a 2-form gauge potential…

Differential Cohomology Theory

In order to describe B we need to enter the world of differential generalized cohomology theories…

If $\mathcal{E}$ is a generalized cohomology theory, then a machine of Hopkins & Singer produces $\check{\mathcal{E}}$

$$0 \to \mathcal{E}^{j-1}(M, \mathbb{R}/\mathbb{Z}) \to \check{\mathcal{E}}^j(M) \to \Omega_{\mathbb{Z}}(M; \mathcal{E}(pt) \otimes \mathbb{R})^j \to 0$$

etc.

Orientation & Integration

The orientation twisting of $\mathcal{E}(M)$ is denoted $\tau^{\mathcal{E}}(TM)$:

It allows us to define an "integration" in $\mathcal{E}$-theory:

$$\int_M^{\mathcal{E}} : \mathcal{E}^{\tau^{\mathcal{E}}(TM)+j}(M) \to \mathcal{E}^j(pt)$$

For the oriented bosonic string B is a local geometric object, e.g. in one model it is a gerbe connection, denoted $\check{\beta} \in \mathrm{ob}\check{H}^3(X)$

Its gauge equivalence class is an element of differential cohomology: $[\check{\beta}] \in \check{H}^3(X)$

For bosonic orientifolds: $\check{\beta} \in \mathrm{ob}\check{H}^{w+3}(X)$

$$\mathcal{A}_B = \exp[2\pi i \int_{\Sigma/S}^{\check{H}} \varphi^*(\check{\beta})] \in \check{H}^1(S)$$

Integration makes sense because $\varphi^*(w) \cong w_1(\Sigma)$

<u>Surprise!! For superstrings: not correct!</u>

Orientifold Superstring Worldsheets

Spin structure α on $\hat{\Sigma}$

Fermi fields $\psi \in \Gamma(S_\alpha^+ \otimes \hat{\pi}^* \varphi^*(TX))$ ...

Path integral: Integrate over $\varphi, \psi, \hat{\Sigma}$

For dim X = 10, the integral over Fermi fields gives a <u>well-defined measure</u> on

$$S = \mathrm{SpinModuli}(\hat{\Sigma}) \times \mathrm{Map}(\Sigma \to X)$$

times two <u>problematic factors</u> …

$$\mathcal{A}_B \cdot \mathrm{Pfaff}(D_{\hat{\Sigma}}(\hat{\pi}^* \varphi^* (TX - 2)))$$

This must be *canonically* a function on S.
But in truth Pfaff is the section of a line bundle:

$$L_\psi \to S$$

23 years ago, Is Singer asked me:
``How do you sum over spin structures in the superstring path integral?''

It's a good question!!

Related: How does the *spacetime* spin structure enter the worldsheet theory?

Pfaffians

Later on we'll need to be more precise

A spin structure α on $\hat{\Sigma}$ determines, locally, a pair of spin structures on Σ of opposite underlying orientation

$\Rightarrow$ L_ψ is flat… $\Rightarrow$ holonomy is computed by η (Bismut-Freed)

Atiyah-Patodi-Singer flat index theorem gives…

$$L_\psi = \int_{\Sigma/S}^{\check{K}O} \check{\delta} \cdot \varphi^*(TX - 2)$$

$$\in \mathrm{ob}\check{K}O^{-2}(S) \to \mathrm{ob}\check{I}^2(S)$$

$\check{I}^2(S)$ Graded line-bundles with connection

$\check{\delta}$: a flat element of differential KO.
heuristically, it measures the difference
between left and right spin structures.

Since Pfaff is problematic,
the B-field amplitude, $\mathcal{A}_B$,
must also be problematic.

What is the superstring B-field anyway !?

How to find the B-field

- To find out where B lives let us turn to the *spacetime* picture.

- The RR field must be formulated in terms of differential K-theory of spacetime.

- The B-field twists that K-theory

- For orientifolds, the proper version of K-theory is $KR(X_w)$ (Witten)

X_w is a "stack" $\Rightarrow$ careful with $KR(X_w)$

For $X = Y /\!/ \Gamma$ use Fredholm model

(Atiyah, Segal, Singer)

$\mathcal{H}$: $\mathbb{Z}_2$-graded Hilbert space with stable Γ-action

Γ_0: Is linear $\qquad$ Γ_1: Is anti-linear

$\mathcal{F}$: Skew-adjoint odd Fredholms

$$KR(X_w) = [T : Y \to \mathcal{F}]^{\Gamma}$$

Assume all goes well for $\check{K}R(X_w)$

Twistings

- We will consider a special class of twistings with geometrical significance.
- We will consider the degree to be a twisting, and we will twist by a ``graded gerbe.''
- The twistings are objects in a groupoid. They are classified topologically by a generalized cohomology theory.
- But to keep things simple, we will systematically mod out by Bott periodicity.

Twisting K (mod Bott)

When working with twistings of K (modulo Bott periodicity) it is useful to introduce a ring theory:

$$R = ko\langle 0 \cdots 4\rangle$$

$$\pi_* R = \mathbb{Z}[\eta, \mu]/(\deg \geq 5)$$

Twistings of K(X) classified by $R^{-1}(X)$

As a set: $H^0(X, \mathbb{Z}_2) \times H^1(X, \mathbb{Z}_2) \times H^3(X, \mathbb{Z})$

Twistings of KR

For twistings of $KR(X_w)$: $R^{w-1}(X)$

$$H^0(X,\mathbb{Z}_2)\times H^1(X,\mathbb{Z}_2)\times H^{w+3}(X,\mathbb{Z})$$

Warning! Group structure is nontrivial, e.g.:

$$R^{w-1}(pt/\!/\mathbb{Z}_2)\cong\mathbb{Z}_8$$

Reflecting Bott-periodicity of KR.

(Choose a generator θ for later use.)

The Orientifold B-field

So, the B-field is a geometric object whose gauge equivalence class (modulo Bott) is

$$[\check{\beta}] \in \check{R}^{w-1}(X)$$

Topologically: $[\beta] = (t, a, h) \in H^0 \times H^1 \times H^{w+3}$

t=0,1: IIB vs. IIA.

a: Related to $(-1)^F$ & Scherk-Schwarz

h: is standard

The RR field is self-dual

We conclude from the above that the RR ***current*** is

$$\check{j}_{RR} \in \check{K}R^{\check{\beta}}(X_w)$$

But self-duality imposes restrictions on the B-field

We draw on the Hopkins-Singer paper which, following Witten, shows that a central ingredient in a self-dual abelian gauge theory is a ***quadratic refinement*** of the natural pairing of electric and magnetic currents…

Quadratic functor hierarchy

In fact, the HS theory produces compatible quadratic functors in several dimensions with different physical interpretations:

dim=12 $$q : \check{K}R(M) \to \mathbb{Z}$$

dim=11 $$q : \check{K}R(N) \to \mathbb{R}/\mathbb{Z}$$

dim=10 $$q : \check{K}R(X) \to \check{I}^2(pt)$$

(In families over T: Map to $\check{I}^{0,1,2}(T)$)

Parenthetical Remark: Holography

If spacetime X is the boundary of an 11-fold N: $\partial N = X$

Then we may view $\check{j}$ on X as the *boundary value* of a gauge field on N.

Self-dual gauge theory on X is holographically dual to Chern-Simons gauge theory on N with action $q(\check{j})$.

Defining our quadratic function

Basic idea is that we want a formula of the shape

$$q(j) = \int_M^{KO} \bar{j} j \in \mathbb{Z}$$

How to make sense of it?

$$j \to \bar{j} j \in KR^{\bar{\beta}+\beta}(M_w)$$ is ``real''

$\bar{\beta} + \beta$ induces a twisting $\Re(\beta)$ of $KO_{\mathbb{Z}_2}$

$$j \to \bar{j} j$$

$$KR^{\beta}(M_w) \to KO_{\mathbb{Z}_2}^{\Re(\beta)}(M_w)$$

But, to integrate, we need:

$$\boxed{KO_{\mathbb{Z}_2}^{\Re(\beta)}(M_w) \to KO_{\mathbb{Z}_2}^{\tau^{KO_{\mathbb{Z}_2}}(TM_w-12)}(M_w)}$$

$$KO_{\mathbb{Z}_2}^{\tau^{KO_{\mathbb{Z}_2}}(TM_w-12)}(M_w) \xrightarrow{\int^{KO_{\mathbb{Z}_2}}} KO_{\mathbb{Z}_2}^{-12}(pt)$$

$$KO_{\mathbb{Z}_2}^{-12}(pt) \cong \mathbb{Z} \oplus \varepsilon\mathbb{Z} \to \varepsilon\mathbb{Z}$$

Twisted Spin Structure

The ***twisted spin structure*** is an isomorphism of KO_{Z2}-twistings

$$\kappa : \Re(\beta) \to \tau^{KO_{\mathbb{Z}_2}}(TM_w - 12)$$

Note: A spin structure on M allows us to integrate in KO. It is an isomorphism

$$0 \to \tau^{KO}(TM - 12)$$

One corollary of the existence of a twisted spin structure is a constraint relating the topological class of the B-field (mod Bott) to the topology of X

$$[\beta] = (t, a, h) \in H^0 \times H^1 \times H^{w+3}$$

$$w_1(X) = tw$$

$$w_2(X) = tw^2 + aw$$

(The quadratic function also allows us to define the RR charge of ``orientifold planes.''
I will return to this at the end.)

Examples

Zero B-field

If [β]=0 then we must have IIB theory on X which is orientable and spin.

Op-planes

$$X = \mathbb{R}^{p+1} \times \mathbb{R}^r /\!\!/ \mathbb{Z}_2 \qquad p + r = 9$$

Compute: $w_1(X) = rw \qquad w_2(X) = \frac{r(r-1)}{2} w^2$

$$\Rightarrow \quad t = r \mod 2 \qquad a = \begin{cases} 0 & r = 0,3 \mod 4 \\ w & r = 1,2 \mod 4 \end{cases}$$

How to sum over worldsheet spin structures

Now let us return to our difficulty on the ws:

$$\mathcal{A}_B \cdot \mathrm{Pfaff}(D_{\hat{\Sigma}}(\hat{\pi}^* \varphi^*(TX - 2)))$$

must be canonically identified with a function.

$$\mathcal{A}_B \stackrel{?}{=} \exp[2\pi i \int_{\Sigma/S}^{\check{R}} \varphi^*(\check{\beta})]$$

NO! Integrand is not a proper density for integration in R-theory!

The B-line

Orientations in R-theory are induced by orientations in KO, but Σ does not have a spin structure!

Use the class $\check{\delta}$ constructed from the spin structure α on $\hat{\Sigma}$:

$$L_B := \int_{\Sigma/S}^{\check{R}} \check{\delta} \cdot \varphi^*(\check{\theta}\check{\beta})$$
$$\in \mathrm{ob}\check{R}^{-2}(S) \to \mathrm{ob}\check{I}^2(S)$$

Comes with a canonical section which we define to be $\mathcal{A}_B$

We are in the process of proving the following

Theorem: A twisted spin structure κ induces a canonical trivialization of $L_B \otimes L_\psi$

idea of the proof…

Recall that the Pfaffian is a section of

$$L_\psi = \int_{\Sigma/S}^{\check{K}O} \check{\delta} \cdot \varphi^*(TX - 2) \in \mathrm{ob}\check{I}^2(S)$$

and R is a quotient of KO…

Let r classify twistings of KO mod Bott:

$$\pi_*(r) = \mathbb{Z}_8[\eta]/(\eta^3, 2\eta)$$

1. $\int^{KO} \delta \cdot \varphi^*(TX - 2)$ in $I^2(S)$
   only depends on the image of $\varphi^*(TX - 2)$
   in $r^0(\Sigma)$, which is $\varphi^*(\tau^{KO}(TX - 2))$

2. $\int^{R} \delta \cdot \varphi^*(\theta \cdot \beta)$ in $I^2(S)$
   only depends on the image of $\varphi^*(\theta \cdot \beta)$
   in $r^0(\Sigma)$, which is $\varphi^*(\Re(\beta))$

Homework solution (23 years late)

!! Since a twisted spin structure gives
a canonical trivialization of $L_B \otimes L_\psi$

$$\mathcal{A}_B \cdot \mathrm{Pfaff}: \text{ Canonically a function}$$

Therefore, the same datum that allows us to define the RR field in spacetime, also is the key ingredient that leads to anomaly cancellation on the worldsheet.

Some key tests

- w=0: Ordinary type II string. Changing t=0 to t=1 correctly reproduces the expected change in the GSO projection due to the mod-two index of Dirac.

$$[\beta] = (t = 1, 0, 0) \qquad \mathcal{A}_B = (-1)^{\mathrm{mod}_2(\alpha_L)}$$

- A change of <u>*spacetime*</u> spin structure changes the amplitude in the expected way.

RR charge of O-planes

- Components of the fixed-point loci in X_w are known as ``orientifold planes.''
- They carry RR charge
- Mathematically, the charge is the center of the quadratic function: $q(j) = q(2\mu - j)$
- Once we invert 2 we can compute μ using localization of a KO_{Z2}-integral. Then the charge localizes to the O-planes and is:

$$\mu = \frac{1}{2}\iota_* \left(\frac{\kappa^{-1}\Xi(F)}{\psi_{1/2}(\kappa^{-1}\mathrm{Euler}(\nu))} \right)$$

$\iota : F \hookrightarrow X_w$: Inclusion of a component F of the fixed point set with normal bundle ν

$\psi_{1/2}$: Multiplicative inverse of Adams ψ_2

$\Xi(F)$: KO-theoretic Wu class (related to ``Bott's cannibalistic class'')

$$\int_F^{KO} \psi_2(x) = \int_F^{KO} \Xi(F) x$$

The physicists' formula

Taking Chern characters and appropriately normalizing the charge we get the physicist's formula for the charge in de Rham cohomology:

$$-\sqrt{\hat{A}(TX)}\mathrm{ch}(\mu) = \pm 2^{p-4}\iota_*\sqrt{\frac{L'(TF)}{L'(\nu)}}$$

$$L'(V) = \prod_i \frac{x_i/4}{\tanh(x_i/4)}$$

Predicting Solitons (w=0)

p	Magnetic	Electric
0	0	$R^{-1}(pt) = \mathbb{Z}_2$
1	0	$R^{0}(pt) = \mathbb{Z}$
2	0	0
3	0	0
4	0	0
5	$R^{-4}(pt) = \mathbb{Z}$	0
6	0	0
7	$R^{-2}(pt) = \mathbb{Z}_2$	0
8	$R^{-1}(pt) = \mathbb{Z}_2$	0
9	$R^{0}(pt) = \mathbb{Z}$	0

string

5-brane

Orientifold Précis : NSNS Spacetime

1. X: dim=10 Riemannian orbifold with dilaton

2. Orientifold double cover: $w \in H^1(X, \mathbb{Z}_2)$

3. B-field: Geometric twisting of $\check{K}R(X_w)$.

$$\text{mod Bott: } [\check{\beta}] \in \check{R}^{w-1}(X)$$

4. Twisted spin structure:

$$\kappa : \Re(\beta) \to \tau^{KO}(TX - 2)$$

Orientifold Précis: Consequences

1. Well-defined worldsheet measure.

2. K-theoretic definition of RR charge of O-planes

3. Localization formula (inverting $(1-\varepsilon)$)

$$\mu = \frac{1}{2}\iota_* \left(\frac{\kappa^{-1}\Xi(F)}{\psi_{1/2}(\kappa^{-1}\mathrm{Euler}(\nu))} \right)$$

4. Well-defined spacetime fermions,

   and well-defined coupling to RR field

5. Possibly, new solitons

Conclusion

The main future direction is in applications

Destructive String Theory?

- Tadpole constraints (Gauss law)
- Spacetime anomaly cancellation

Thank you Is !

And Happy Birthday !!