Preprint typeset in JHEP style - HYPER VERSION

A Very Long Lecture on the Physical Approach to Donaldson and Seiberg-Witten Invariants of Four-Manifolds

Gregory W. Moore

NHETC and Department of Physics and Astronomy, Rutgers University

gmoore@physics.rutgers.edu

Abstract: These are lecture notes for a lecture at the Simons Center for Geometry and Physics School on Mathematical Applications of Supersymmetric Yang Mills Theory, March 8, 2012

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1. Preamble: Topology of Four-Manifolds

Let us summarize some of the standard results on 4-manifolds. See the texbooks [3, 6, 23] for details.

1.1 Fundamental group

First, if G is any finitely presented group then there is a compact 4-fold X with $\pi_1(X) \cong G$. (Four is the first dimension in which this happens.) Since the word problem for groups is undecidable this means we cannot hope to classify all compact 4-manifolds. But we can still hope to understand simply connected 4-folds.

1.2 Intersection Form

There is another interesting topological invariant, the intersection number. Let

$$\bar{H}^2(X) := H^2(X,\mathbb{Z})/Torsion. \tag{1.1}$$

We will think of it as the image of $H^2(X,\mathbb{Z})$ in $H^2(X;\mathbb{R})$. This is a lattice of rank b_2. Then if X is compact and oriented we have

$$Q_X : \bar{H}^2(X) \times \bar{H}^2(X) \to \mathbb{Z} \tag{1.2}$$

$$Q_X(\omega_1, \omega_2) := \int_X \omega_1 \wedge \omega_2 \tag{1.3}$$

For X compact and oriented this is a perfect pairing by Poincaré duality, so Q_X is a unimodular integral symmetric form. If α is a cohomology class let $S(\alpha)$ be the Poincaré dual cycle:

$$\int_X \alpha\beta = \int_{S(\alpha)} \beta \tag{1.4}$$

Then $Q_X(\alpha, \beta) = S(\alpha) \cdot S(\beta)$ is the oriented intersection number.

1.3 Whitehead theorem

In 1949 J.H.C. Whitehead introduced the notion of CW decomposition of manifolds to classify homotopy type. In [17] Milnor observed that an interesting consequence is that two simply connected oriented four-manifolds X_1, X_2 are homotopy equivalent iff $Q_{X_1} \cong Q_{X_2}$.

1.4 Serre's theorem

Thus we come to the classification of integral unimodular forms. Serre gave a nice classification in the *indefinite* case.

	Indefinite	Definite
even	$mE_8 \oplus nH$,$m \in \mathbb{Z}$, $n > 0$	$1, 2, 24, > 10^7, \ldots$
odd	$m\langle +1\rangle \oplus n\langle -1\rangle$	too many

The even definite forms only exist in dimension 0 modulo 8. We have listed the number of inequivalent ones for the first few cases. The unique lattice in dimension 8 is the E_8 root lattice.

H denotes the even integral form on $\mathbb{Z} \oplus \mathbb{Z}$ given by

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.5}$$

1.5 Freedman's theorem: Homeomorphism type

In the 1980's Michael Freedman achieved a breakthrough:

For all unimodular integral forms Q there is a simply connected compact orientable topological manifold X with $Q \cong Q_X$. Moreover,

1. If Q is even then there is a unique such X up to homeomorphism.

2. If Q is odd then there are exactly two homeomorphism types and at most one of them can be smooth.

As an example of how breathtaking this is note that for $Q = 0$ this proves the (four-dimensional) Poincaré conjecture. For $Q = 1$ we have $X = \mathbb{C}P^2$ but there must be another manifold, "fake $\mathbb{C}P^2$" which is homeomorphic to $\mathbb{C}P^2$ but does not admit a smooth structure!

1.6 Donaldson's Theorems: Diffeomorphism type

Shortly after Freedman's work (c. 1983) Donaldson announced some equally striking theorems.

First, if X admits a smooth structure and Q_X is definite, then it must be diagonal: $m\langle +1\rangle$ or $m\langle -1\rangle$.

Remarkable corollaries include the fact that the manifold corresponding to $2E_8$ does not admit a smooth structure. (All previous known tests - notably Rokhlin's theorem - admitted the possibility that it might.)

Second, Donaldson introduced his famous polynomial invariants. These are a sequence of polynomial function on $H_0(X) \oplus H_2(X)$ which are invariants of the *smooth structure* of X.

Donaldson's construction used nonabelian gauge theory (for $G = SU(2)$) and in particular defines the polynomials using the intersection theory on the moduli space of anti-self-dual connections on principal G bundles over X. Now, the equation

$$F^+ = F + *F = 0 \tag{1.6}$$

makes use of a Riemannian metric. But the dependence on the metric drops out *except* for manifolds with $b_2^+ = 1$. The metric dependence in the case $b_2^+ = 1$ is completely understood. We will come back to these important facts.

Donaldson's invariants were used to prove some striking facts about the smooth structures of 4-manifolds.

After all this progress – Freedman and Donaldson both received the 1986 Fields medal – it was natural to wonder if physics was playing an important role. After all, Donaldson was using nonabelian gauge theory!

This is where Witten enters. In 1988 he gave a quantum field theoretic description of Donaldson polynomials [25]. We will describe it in detail (in part following a particularly beautiful approach to Witten's paper introduced by M. Atiyah and L. Jeffrey [1]).

Witten's interpretation was beautiful - it was the genesis of the concept of topological twisting and more broadly of topological field theory - but it was not clear what could be gained mathematically from an interpretation of the Donaldson polynomials in terms of a path integral. The problem is that the path integral of a four-dimensional interacting quantum field theory is regarded by the mathematical community as a mythological being.

However, since the theory is topological, the path integral can be recast in terms of an *effective* theory of (arbitrarily) low energy fluctuations above the vacuum. This means that with a sufficiently good understanding of the quantum vacua of the theory one can hope to recast the Donaldson-Witten path integral in a new form which might yield new insights.

This is precisely what happened. In the spring of 1994 Seiberg and Witten understood the vacuum structure of N=2 SU(2) SYM on $\mathbb{R}^4$ [22, 21]. This was sufficient information for Witten to give a stunning reformulation of the Donaldson polynomials [27].

The goal of this lecture is to explain Witten's formal field theory interpretation of Donaldson's polynomials and then to show how Seiberg and Witten's physical insights into the dynamics of $N = 2$ SYM lead to a compelling reformulation of the Donaldson polynomials.

Most of the material for this lecture can be found in the very nice textbook [9].

2. Plan for the Rest of the Talk

1. Formal structure of cohomological TFT: Mathai-Quillen form of the path integral and localization.
2. How Donaldson Theory fits into the MQ framework: Twisted N=2 SYM.
3. SW solution and structure of the vacuum: Mapping observables from UV to IR.
4. General form of the Higgs branch contribution.
5. Evaluation of the Coulomb branch: The u-plane integral.
6. Deriving the relation of Donaldson to SW invariants.
7. Simple type
8. Applications of the physical viewpoint: (Expample: Superconformal simple type and the generalized Noether inequality.)
9. Possible future directions.

3. Part I: TFT Integrals and MQ form of the twisted N=2 theory.

3.1 A nice integral

Today we are going to talk about some fancy integrals, but let us start with a very simple one. Let ϕ be a real number and consider a function $s(\phi)$ whose graph is transverse to the ϕ axis.

FIGURE

Consider the Gaussian integral:

$$Z = \int_{-\infty}^{+\infty} \frac{d\phi}{\sqrt{2\pi}} s'(\phi) e^{-\frac{1}{2}(s(\phi))^2} \tag{3.1}$$

It is easy to do the integral by change of variable, and you find:

$$Z = \sum_{\phi_i : s(\phi_i)=0} \frac{s'(\phi_i)}{|s'(\phi_i)|} \tag{3.2}$$

It is easy to generalize to n-dimensions. Now $s : \mathbb{R}^n \to \mathbb{R}^n$ and:

$$Z = \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{d\phi_i}{\sqrt{2\pi}} \det(\frac{\partial s^i}{\partial \phi^j}) e^{-\frac{1}{2}(s(\phi),s(\phi))} = \sum_{s(\phi)=0} \mathrm{sign}(\det \frac{\partial s^i}{\partial \phi^j}) \tag{3.3}$$

where we use the Euclidean metric on $\mathbb{R}^n$ to define $(s(\phi), s(\phi))$.

Let us make a few remarks about these integrals

1. The answer is a sum of integers. It is a *signed* sum over solutions of the n real equations in n unknowns:

$$s(\phi) = 0 \tag{3.4}$$

Our integral Z is counting solutions to equations with signs.

2. In fact, this integer has topological significance. It is the *degree* of the (proper) map $s : \mathbb{R}^n \to \mathbb{R}^n$. Another topological interpretation is that it is the oriented intersection number of the graph of s with the graph of $s = 0$.

3. Finally, note that we could put in a parameter $\hbar$ and equally well say:

$$Z = \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{d\phi_i}{\sqrt{2\pi\hbar}} \det(\frac{\partial s^i}{\partial \phi^j}) e^{-\frac{1}{2\hbar}(s(\phi),s(\phi))} = \sum_{s(\phi)=0} \text{sign}(\det \frac{\partial s^i}{\partial \phi^j}) = \deg(s) \tag{3.5}$$

The answer is independent of $\hbar$. On the other hand, we could take $\hbar \to 0$ and clearly the measure *localizes* to the zero set

$$\mathcal{Z}(s) := \{\phi : s(\phi) = 0\} \tag{3.6}$$

Moreover, the saddle-point approximation gives the exact answer.

How do we explain all this? Supersymmetry!

3.2 Supersymmetric Representation of the nice integral

3.2.1 Superspace

We are going to rewrite this integral in a form that makes contact with topologically twisted path integrals.

First, if M is any manifold, there is an associated *superspace* $\widehat{M}$ given by

$$\widehat{M} = \Pi T^* M \tag{3.7}$$

If ϕ^i are local coordinates then ψ^i are corresponding odd fiber coordinates.

We have the key isomorphism

$$\mathcal{C}^\infty(\widehat{M}) \cong \Omega^*(M) \tag{3.8}$$

where

1. $\psi^i \leftrightarrow d\phi^i$

2. There is an integral grading so that "*ghost number*" corresponds to degree of the differential form:

$$\text{gh}\#(\hat{\omega}) = \deg(\omega) \tag{3.9}$$

3. There is a degree one derivation which squares to zero:

$$Q\hat{\omega} \leftrightarrow d\omega \tag{3.10}$$

4. There is a Berezinian on the superspace so that

$$\int_{\widehat{M}} \hat{\mu}\hat{\omega} = \int_M \omega \tag{3.11}$$

With proper orientations

$$\hat{\mu} = \prod_{i=1}^{n} d\phi^i d\psi^i \tag{3.12}$$

Note particularly that:

$$Q\phi^i = \psi^i \qquad Q\psi^i = 0 \tag{3.13}$$

3.2.2 Rewriting the integral

Now we introduce anticommuting variables χ_a, where, for the moment, $a = 1, \ldots, n$ and rewrite the integral as

$$Z = \xi i^n \int_{\widehat{\mathbb{R}^n}} \prod_{i=1}^{n} \frac{d\phi^i d\psi^i}{\sqrt{2\pi\hbar}} \int_{\Pi(\mathbb{R}^n)^*} \prod_{a=1}^{n} d\chi_a e^{-\frac{1}{2\hbar}s^a(\phi)s^a(\phi)+i\chi_a \frac{ds^a}{d\phi^j}\psi^j} \tag{3.14}$$

We make one further maneuver of introducing a simple Gaussian integral over some commuting coordinates H_a to make some things manifest:

$$Z = \xi \int_{\widehat{\mathbb{R}^n}} \hat{\mu} \int_{\widehat{(\mathbb{R}^n)^*}} \prod_{a=1}^{n} \frac{dH_a d\chi_a}{(2\pi i)} e^{-\frac{\hbar}{2}H_a H_a - iH_a s^a + i\chi_a \frac{ds^a}{d\phi^j}\psi^j} \tag{3.15}$$

where $\xi = \pm 1$ depends on how we orient the $d\psi d\chi$.

Now we have it where we want it:

1. If we extend Q so that

$$Q\chi_a = H_a \qquad QH_a = 0 \tag{3.16}$$

then we can write the "action" S (so the path integral contains e^S) as:

$$S = Q(\Psi) \tag{3.17}$$

$$\Psi = -\frac{\hbar}{2}\chi_a H_a - i\chi_a s^a \tag{3.18}$$

2. Now note that since $Q \leftrightarrow d$ if we change the action by $Q(\Delta\Psi)$ so that the integral over the boundary (in this case, the integral at infinity) vanishes, then the integral is unchanged. That is

Small Q-exact perturbations of the action do not change the result of the integral.

3. In cohomological field theory, (χ_a, H_a) is called the *anti-ghost multiplet*. Note that χ_a has ghost number -1, so H_a has ghost number 0.

4. Note that evaluation of the Gaussian integral on H_a gives $H_a = -is^a/\hbar$ so that $\psi = \chi = 0$ and $s^a = 0$ are the Q-fixed points in fieldspace. It turns out to be very useful to identify the Q-fixed points with $\mathcal{Z}(s)$. Intuitively, since $\int d\theta = 0$ for a Grassmann integral it is natural for a supersymmetric integral to localize to Q-fixed points.

3.2.3 Generalization to "nonzero index"

We can generalize a little by letting $s : \mathbb{R}^n \to V$ where $V \cong \mathbb{R}^m$ has dimension m not necessarily equal to n. Now we let the anti-ghost multiplet (χ_a, H_a) be indexed by $a = 1, \dots, m$.

Define:

$$\widehat{\mathrm{Eul}}_s := \int_{\widehat{V}} \prod \frac{d\chi_a dH_a}{(2\pi i)} e^{Q(\Psi)} \tag{3.19}$$

This is BRST closed and hence represents a closed differential form Eul_s via the correspondence (3.8). Importantly, it has ghost number m.

Now, if $\hat{\mathcal{O}}(\phi, \psi)$ is another BRST-closed observable, then we can consider the more general integral

$$\int_{\widehat{\mathbb{R}^n}} \hat{\mu} \;\; \hat{\mathcal{O}} \;\; \widehat{\mathrm{Eul}}_s \tag{3.20}$$

Note that

1. This vanishes unless the ghost number of $\hat{\mathcal{O}}$ is $n - m$.
2. The integral does not depend on the precise BRST representative of $\hat{\mathcal{O}}$.

Moreover, $\hat{\mathcal{O}}$ corresponds to a closed form $\mathcal{O}$ on $\mathbb{R}^n$ and we can generalize what we said above by saying that

$$\int_{\widehat{\mathbb{R}^n}} \hat{\mu} \hat{\mathcal{O}} \widehat{\mathrm{Eul}}_s = \int_{\mathbb{R}^n} \mathcal{O} \wedge \mathrm{Eul}_s = \int_{\mathcal{Z}(s)} \mathcal{O} \tag{3.21}$$

localizes on the zero set $\mathcal{Z}(s)$.

3.3 Thom Isomorphism Theorem

We are going to generalize our integral further. Let $\pi : E \to M$ be an oriented vector bundle over an n-dimensional manifold, where E has rank m. Then the Thom isomorphism theorem says there is an isomorphism

$$H^i(M) \cong H^{i+m}_{v-cpt}(E) \tag{3.22}$$

$$\omega \to \pi^*(\omega)\Phi(E) \tag{3.23}$$

Moreover if $s : M \to E$ is a section then $s^*\Phi(E)$ is the *Euler class of* E. If M is compact the Euler class is Poincaré dual to the zero set of s:

$$\int_M \omega s^*(\Phi(E)) = \int_{\mathcal{Z}(s)} \omega \tag{3.24}$$

This motivates us to generalize $\widehat{\mathrm{Eul}}_s$ above to the case where we replace $\mathbb{R}^n \times V$ by an oriented vector bundle $E \to M$. To do this we must give E a connection ∇. We will denote the local one-form by Θ_j^{ab}. To covariantize the action we must add a third term to Ψ:

$$\Psi = -\frac{\hbar}{2}\chi_a H_a - i\chi_a s^a + \frac{\hbar}{2}\chi_a \Theta_i^{ab} \psi^i \chi_b \tag{3.25}$$

Working out $Q\Psi$ the third term covariantizes the derivative of s, and integrating out the auxiliary fields H_a we find

$$S = -\frac{1}{2\hbar} s^a s^a + i\chi_a (\nabla_j s)^a \psi^j + \frac{\hbar}{4}\chi_a \chi_b F^{ab}_{ij} \psi^i \psi^j \tag{3.26}$$

3.4 The localization formula

Now, the general localization formula is the following:

Define

$$\widehat{\mathrm{Eul}}_s(E, \nabla) := \int \prod_{a=1}^{m} d\chi_a e^S \tag{3.27}$$

where S is given by (3.26).

The connection ∇ on E defines a linear operator:

$$\nabla s : T_p M \to E_p$$

Therefore, we can form the bundle $\mathrm{Cok}\nabla s$ given by the cokernel of this operator:

$$0 \to \mathrm{Im}\nabla s \to E \to \mathrm{Cok}\nabla s \to 0. \tag{3.28}$$

This bundle will be oriented and we claim that

$$\int_{\hat{E}} \hat{\mu} e^S \hat{\mathcal{O}} = \int_{\widehat{M}} \hat{\mu} \widehat{\mathrm{Eul}}_s(E, \nabla) \hat{\mathcal{O}} = \int_{\mathcal{Z}(s)} \mathcal{O} \wedge \mathrm{Eul}(\mathrm{Cok}\nabla s) \tag{3.29}$$

where $\hat{E}$ is the total superspace corresponding to the bundle E.

1. The proof is straightforward.
2. It is possible to view $\widehat{\mathrm{Eul}}_s(E, \nabla)$ as the pullback by s of a representative of a Thom class $\hat{\Phi}(E)$. This particular representative of the Thom class is due to Mathai and Quillen. Note that it has rapid decrease along the fibers rather than compact support. For a full explanation see [2].

3.4.1 Equivariance

For applications to gauge theories we need one more formal development. Suppose that M is a $\mathcal{G}$-space for some Lie group $\mathcal{G}$ and that $E \to M$ is a $\mathcal{G}$-equivariant vector bundle with $\mathcal{G}$-equivariant connection and covariant section s, so that $\mathcal{G}$ acts on the zero-set $\mathcal{Z}(s)$.

Then, one needs to introduce another supersymmetric multiplet, and another term in Ψ and the localization formula becomes instead:

$$\int_{\hat{E}'} \hat{\mu} e^S \hat{\mathcal{O}} = \int_{\mathcal{M}} \iota^*(\mathcal{O}) \wedge \mathrm{Eul}(\mathrm{Cok}(\mathbb{O})/\mathcal{G}) \tag{3.30}$$

for gauge invariant BRST closed $\hat{\mathcal{O}}$. Here $\mathbb{O} = \nabla s \oplus \cdots$ where the extra term has to do with handling equivariance, and

$$\mathcal{M} := \mathcal{Z}(s)/\mathcal{G} \tag{3.31}$$

is the *moduli space of solutions to the equation* $s(\phi) = 0$.

We will skip the detailed discussion of the equivariant generalization. For details see [2] for an extended and leisurely discussion and [18] for a lightning summary.

3.5 The Fields, Equations, Symmetries Paradigm

Now, the point of all the above formal development is this:

All topologically twisted quantum field theories fit in the above paradigm.

Quite generally, to specify a topological field theory in what we will call the "Mathai-Quillen" form one needs to specify

1. *Fields*: These are represented by the ϕ^i. These might be maps of a surface into a target space, or connections on a principal bundle.

2. *Equations*: We are interested in some equations on the fields $s(\phi^i)$. They are generally interesting partial differential equations. These might be the equations determining whether a map is a pseudoholomorphic map, or whether a connection is a Yang-Mills instanton.

3. *Symmetries*: Typically the equations have gauge symmetry.

The main statement, as above, is that the path integral localizes to the moduli space

$$\mathcal{M} := \{\phi : s(\phi) = 0\}/\mathcal{G} \tag{3.32}$$

and, if we include operator insertions, the path integral computes integrals of cohomology classes over this moduli space.

The linear operator ∇s we encountered above will be a Fredholm operator, typically associated with an elliptic complex related to the equation. When its index is nonzero then we have an analog of the situation $n - m \neq 0$ and we will need to insert operators with the appropriate ghost number in order to get a nonzero path integral.

The basic paradigm here is due to Witten [Cite:ICTP Lectures]. The reference [2] works out in detail the MQ formalism for many of the popular cohomological topological field theories.

4. Twisted N=2 SYM in Mathai-Quillen Form

4.1 Fields, Equations, Symmetries

The basic data:

1. A closed, oriented, Riemannian 4-manifold $(X, g_{\mu\nu})$.

2. A principal bundle $P \to X$ for a compact Lie group G with Lie algebra $\mathfrak{g}$.

Our fields ϕ^i will be $A \in \mathcal{A} := \mathrm{Conn}(P)$.

Our bundle of equations will be

$$\mathcal{E} = \mathcal{A} \times \Omega^{2,+}(X, \mathrm{ad}P) \tag{4.1}$$

and our section will be

$$s(A) := F^+ := F + *F \tag{4.2}$$

This is equivariant for the group of symmetries:

$$\mathcal{G} = \mathrm{Aut}(P) \sim \mathrm{Map}(X, G) \tag{4.3}$$

We can now run the MQ machine. The path integral localizes to the moduli space of instantons $\mathcal{M}(P, g)$.

4.2 Relation to twisted $N = 2$ SYM

4.2.1 The standard $N = 2$ VM

Now we consider N=2 SYM in 4 dimensions. In Euclidean signature it has "Lorentz" symmetry $su(2)_- \oplus su(2)_+$, "R-symmetry" $su(2)_R \oplus u(1)_R$, and gauge symmetry $\mathfrak{g}$.

The standard N=2 VM is recorded in the following table:

	$su(2)_- \oplus su(2)_+ \oplus su(2)_R$	$u(1)_R$	$\mathfrak{g}$
A_μ	$(2,2,1)$	0	$\mathfrak{g}$
$\bar{\psi}^A_{\dot{\alpha}}$	$(1,2,2)$	-1	$\mathfrak{g}$
ψ^A_α	$(2,1,2)$	1	$\mathfrak{g}$
ϕ	$(1,1,1)$	2	$\mathfrak{g} \otimes \mathbb{C}$
$\bar{\phi}$	$(1,1,1)$	-2	$\mathfrak{g} \otimes \mathbb{C}$
D	$(1,1,3)$	0	$\mathfrak{g}$
$\bar{Q}^A_{\dot{\alpha}}$	$(1,2,2)$	1	1
Q^A_α	$(2,1,2)$	-1	1

1. I have used mathematicians' notation for representations of $su(2)$, denoting them by their *dimension*.

2. The $u(1)_R$ quantum number will correspond to "ghost number," which in turn will correspond to differential form degree on the moduli space of instantons.

4.2.2 Topological twisting

Now one of the key innovations of Witten's 1988 paper was the concept of topological twisting.

One way to say it is that we change the coupling to gravity by redefining the Lorentz group so that

$$su(2)'_+ = \mathrm{Diag} \subset su(2)_+ \oplus su(2)_R \tag{4.4}$$

This has the practical consequence that we read off the geometrical interpretation of the fields by taking the tensor product of the appropriate representations in the above table.

A more conceptual way to say this is that we gauge the $SU(2)_R$-symmetry and choose a connection on that bundle to be identically equal to the self-dual part of the spin connection. When that is done there are remarkable cancelations in the path integral accounting for topological invariance.

The key motivation for this topological twisting is that we want a supersymmetry operator which can function as a BRST operator. That means:

1. It must square to zero.
2. It must be a scalar, and so it can be defined on arbitrary Riemannian 4-folds.

This motivates the twisting defined above, and we take

$$Q = \delta_A^{\dot\alpha} \bar Q^A_{\dot\alpha}. \tag{4.5}$$

Now, given the topological twisting we can recognize the fields in the MQ description:

1. A_μ remains a connection. But from the field ψ^A_α we get an odd 1-form ψ_μ, and $QA_\mu = \psi_\mu$. These correspond to the MQ field multiplets (ϕ^i, ψ^i).

2. From $\bar\psi^A_{\dot\alpha}$ we get an odd self-dual form $\chi_{\mu\nu}$. This is the same χ_a we had above. From $D_{\mu\nu}$ we get an even self-dual form of ghost number 0. This is - essentially - the field H_a we had above.

3. There are also extra fields. From the field $\bar\psi^A_{\dot\alpha}$ we also get an odd zeroform η of ghost number -1. Moreover, we have $\bar\phi$ and ϕ. These have to do with taking into account the equivariance under the group of gauge transformations.

Now having transcribed the fields in this way we find that with $s(A) = F^+$

$$S_{N=2SYM} = Q(\Psi) \qquad ! \tag{4.6}$$

1. Note in particular that changes in the metric $g_{\mu\nu}$ lead to Q-exact perturbations of the action. Thus, the energy-momentum tensor $T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\}$. This is the basis for believing the theory should be topological.

2. As we have seen, the locus $\mathcal{Z}(s)$ can be identified with the Q-fixed points when we study the locus $Q(\text{odd fields}) = 0$ and then put the odd variables to zero. Now,

$$Q\chi_{\mu\nu} = i(F^+_{\mu\nu} - D^+_{\mu\nu}) \tag{4.7}$$

Solving for the auxiliary field gives $D^+_{\mu\nu} = 0$ and hence the Q-fixed points are the anti-self-dual connections.

4.3 Observables

It will turn out that we are in the situation where ∇s has a nonzero index. In this sense the $u(1)_R$ symmetry is anomalous quantum-mechanically. Its anomaly is the virtual dimension of moduli space

$$\mathrm{vdim}\mathcal{M}(P, g) = 4hk - \dim G(b_2^+ - b_1 + 1) \tag{4.8}$$

where h is the dual Coxeter number of $\mathfrak{g}$. In particular, for $SU(2)$,

$$\mathrm{vdim}\mathcal{M}(P, g) = 8k - 3(b_2^+ - b_1 + 1) \tag{4.9}$$

To get an interesting path integral we will need to insert BRST-invariant observables. A crucial point is that

$$Q\phi = 0. \tag{4.10}$$

Now, ϕ is adjoint-valued, so not gauge invariant, but any invariant polynomial of ϕ provides a BRST-invariant and gauge invariant observable. We will call these 0-observables, because they are local operators defined at *points*.

For example, for $SU(N)$ we have independent observables:

$$\mathcal{O}_s^{(0)}(\wp) := \mathrm{Tr}\phi^s(\wp) \qquad s = 2, \ldots, N \tag{4.11}$$

We will mostly be concentrating on the rank one group $SU(2)$ in what follows so we just have

$$\mathcal{O}^{(0)}(\wp) \sim \mathrm{Tr}\phi^2(\wp) \tag{4.12}$$

Remarks

1. Note that $\mathcal{O}^{(0)}(\wp)$ has ghost number 4.
2. At this point we adopt the following policy. To keep equations readable we will suppress real coefficients in some equations. They are typically (fractional) powers of 2 and π. When we do this we use the symbol $\sim$. When I write "=" I really mean "equals." The full expressions with correct coefficients can be found in [19].

Now there is a hierarchy of nonlocal observables. These can be canonically constructed by noting that under topological twisting $Q_{\alpha\dot{\alpha}} \to K_\mu$ with

$$\{Q, K_\mu\} = \partial_\mu \tag{4.13}$$

Therefore if we define $\mathcal{O}^{(1)} := K\mathcal{O}^{(0)}$ we get a 1-form and for a 1-chain γ

$$Q \int_\gamma \mathcal{O}^{(1)} = \mathcal{O}^{(0)}|_{\partial\gamma} \tag{4.14}$$

This implies:

1. A change of location of the point $\wp$ in $\mathcal{O}^{(0)}$ is Q-exact, so we henceforth drop this from the notation and just write $\mathcal{O}$ for the 0-observable.
2. If γ is a closed cycle then $\mathcal{O}(\gamma) := \int_\gamma \mathcal{O}^{(1)}$ is BRST closed.

Similarly, $\mathcal{O}^{(j)} := K^j\mathcal{O}^{(0)}$ define j-forms and if Σ_j is a closed j-cycle then

$$\mathcal{O}(\Sigma_j) := \int_{\Sigma_j} \mathcal{O}^{(j)} \tag{4.15}$$

is a BRST invariant observable which only depends on the homology class of Σ_j. We call these the "j-observables."

Of particular importance in the rank one topologically twisted SYM are the two-observables, which work out to be

$$\mathcal{O}(S) \sim \int_S \mathrm{Tr}(\phi F + \psi \wedge \psi) \tag{4.16}$$

4.4 The Donaldson Polynomials

Donaldson constructed a linear map

$$H_j(X) \to H^{d-j}(\mathcal{M}) \tag{4.17}$$

from the homology of X to the cohomology of $\mathcal{M}$.

Technically, we consider the principal G bundle $P \times \mathcal{A}/\mathcal{G} \to (P \times \mathcal{A})/(G \times \mathcal{G})$. This must have a classifying map

$$f : (P \times \mathcal{A})/(G \times \mathcal{G}) \to BG \tag{4.18}$$

If we choose any $\varpi \in H^d(BG)$ then

$$\omega_D(\Sigma_j) := \int_{\Sigma_j} f^*(\varpi) \in H^{d-j}(\mathcal{A}/\mathcal{G}) \tag{4.19}$$

is a cohomology class which can be restricted to $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$, and it only depends on the homology class of Σ_j.

For $SU(2)$ we choose ϖ to be a generator of $H^4(BSU(2);\mathbb{Z}) \cong \mathbb{Z}$ and thereby define forms:

$\wp \to \omega_D(\wp) \in H^4(\mathcal{M})$

$S \to \omega_D(S) \in H^2(\mathcal{M})$

Now Donaldson defines his polynomials on $H_0(X) \oplus H_2(X)$ by giving the value on the monomial $\wp^\ell S^r$ as

$$P_D(\wp^\ell S^r) := \int_{\mathcal{M}} \omega_D(\wp)^\ell \omega_D(S)^r \tag{4.20}$$

That is, the coefficients are given by intersection numbers on moduli space.

An important point is that $P_D(\wp^\ell S^r)$

1. Are independent of the metric, except for X such that $b_2^+(X) = 1$.
2. Therefore define invariants of the smooth structure of X.

Now, the main claim in Witten's 1988 paper is that *the* 0- *and* 2- *observables precisely correspond to Donaldson's forms* $\omega_D(\wp)$ *and* $\omega_D(S)$, *and hence the generating function of Donaldson polynomials is the twisted SYM path integral with operator insertion*:

$$Z^\xi_{DW}(p,S) := \left\langle e^{p\mathcal{O}+q\mathcal{O}(S)} \right\rangle_{\text{Twisted} \quad \text{N=2} \quad \text{SYM}} = \frac{1}{2}\sum_{\ell,r\geq 0} \frac{(\frac{1}{2}p)^\ell q^r}{\ell! r!} P_D(\wp^\ell S^r) \tag{4.21}$$

1. Here we have taken the gauge group to be $SU(2)$, but since the fields are in the adjoint representation we can take a "twisted $SU(2)$ bundle," that is, an $SO(3)$ bundle which does not lift to an $SU(2)$ bundle. A principal $SO(3)$ bundle over X has two characteristic classes, $\xi = w_2(P) \in H^2(X;\mathbb{Z}_2)$, which we will refer to as the 't Hooft flux, and the instanton number $k \in H^4(X;\mathbb{Z}) \cong \mathbb{Z}$. In the generating function we sum over the instanton number, as appropriate for a path integral. We will refer to $Z^\xi_{DW}(p,S)$ as the *Donaldson-Witten partition function.* (In a physical $SO(3)$ gauge theory, on a compact manifold X, it would also be natural to sum over $\xi \in H^2(X;\mathbb{Z}_2)$.)

2. Of course, when X is not simply connected we can extend this to include 1-observables.

3. The overall factor of $\frac{1}{2}$ is due to the fact that physicists divide by the order of the center of $SU(2)$, which does not act effectively on the fields. The factor of one half in $(\frac{1}{2}p)^\ell$ is a matter of the normalization of $\mathcal{O}$, and we have chosen one to make the physical expressions simpler. Henceforth we absorb q into S.

4. One can give a precise argument relating $\mathcal{O}$ and $\mathcal{O}(S)$ to $\omega_D(\wp)$ and $\omega_D(S)$ following a discussion of Baulieu and Singer. It uses a model for the $\mathcal{G}$-equivariant cohomology of $\mathcal{A}$ and a universal connection. See [2], §8.8 for details.

4.5 Including Hypermultiplets

Now, in N=2 SYM theory it is possible to include another kind of field multiplet, known as a *hypermultiplet*.

This is defined by choosing a quaternionic representation W of G. Choosing a complex structure we may write

$$W = R \oplus R^* \tag{4.22}$$

where R is a complex representation of G.

In a HM the $SU(2)_R$ symmetry acts as the unit quaternions, commuting with the G action. There is a pair of scalar fields $(q, \tilde{q})$ transforming in the $R \oplus R^*$ of G so that $M = (q, \tilde{q}^*)$ is a doublet of $SU(2)_R$. Therefore, when topologically twisted, the scalar fields become a pair of spinors:

$$M \in \Gamma(S^+ \otimes \mathfrak{R}) \qquad \bar{M} \in \Gamma(S^+ \otimes \mathfrak{R}^*) \tag{4.23}$$

where $\mathfrak{R} \to X$ is now a vector bundle associated to $P \to X$ by the representation R.

This raises an important issue: It might well happen that X is not spin, $w_2(X) \neq 0$. In order for the twisted theory to make sense we must take $w_2(\mathfrak{R}) = w_2(X)$ which might require us to choose a certain 't Hooft flux ξ for P.

In the case where P is a $U(1)$ bundle - i.e. in abelian gauge theory we must choose a Spin-c structure.

The way the physicists say this is that we choose a line bundle L^2 which only has a square-root locally, but L does not exist globally. We require that the first Chern class satisfy

$$w_2(X) = c_1(L^2) \mathrm{mod} 2 \tag{4.24}$$

and then take $M \in \Gamma(S^+ \otimes L)$. Neither S^+ nor L exist globally because of -1-signs in the cocycle relation for the transition functions, but the product does exist as an honest vector bundle.

In any case, *when including charged hypermultiplets in an abelian gauge theory we need to introduce a spin-c structure on X.* This will be important later. We will identify a spin-c structure with a class in rational cohomology:

$$\lambda \in \Gamma_w := \frac{1}{2}\bar{w}_2(X) + \bar{H}^2(X) \tag{4.25}$$

where $\bar{w}_2(X)$ is an integral lift of $w_2(X)$.

A few closing remarks on this section:

1. The D-term in the presence of the hypermultiplet turns out to be the quaternionic moment map $\mu(\bar{M}M)^+$ for the action of G on W. Recall now that $\delta\chi = i(F^+ - D^+)$. Together with the variation of fermions in the HM, the Q-fixed point equations work out to be

$$\begin{aligned} F^+ &= \mu(\bar{M}M)^+ \\ \gamma \cdot DM &= 0 \end{aligned} \tag{4.26}$$

   These are known as the *generalized monopole equations*. For the $U(1)$ case, they are the famous *Seiberg-Witten equations* associated to a spin-c structure.

2. At least at a formal level the Donaldson polynomials can be generalized to intersection theory on the moduli spaces of the generalized monopole equations, and can be further generalized to arbitrary compact Lie groups. (The mathematical difficulties with this generalization can be severe: The moduli spaces can be more singular, and can be noncompact. Even for $SU(N)$ with no hypermultiplets one does not have an analog of the "generic metrics theorem" of Freed-Uhlenbeck. Nevertheless, in [8] Kronheimer gave a definition of the $SU(N)$ invariants for all N.)

3. At least at the formal level we fully expect Witten's basic identity (4.21) to hold. Of course, there will be many more BRST invariant observables in this general case and Witten's identity generalizes in an obvious way.

4.6 So, what good is it?

Witten's 1988 paper introduced the idea of a topological field theory and in particular the idea of a topological twisting. This led to a beautiful quantum-field-theory interpretation of Donaldson's polynomials.

With hindsight the interpretation naturally suggests the Seiberg-Witten equations as a natural outcome of the QFT approach, simply because it is natural to couple the VM to HM's. More generally, it suggests generalizations of Donaldson's polynomials to the case of the generalized monopole equations. But that is not how history played out.

In the years following 1988 people asked: "But does the interpretation actually lead to an effective way of evaluating the Donaldson polynomials?" This was not at all clear and several naysayers took a negative attitude, until the fall of 1994....

5. Mapping the UV theory to the IR theory

5.1 Motivation for studying vacuum structure

For topological invariant correlation functions the partition function $Z^{\xi}_{DW}(p,S)$ - which is defined by the UV path integral - should be computable in terms of a low energy effective action:

$$Z^{\xi}_{DW}(p,S) = \left\langle e^{p\mathcal{O}(\wp)+\mathcal{O}(S)} \right\rangle_{UV} = \left\langle e^{p\mathcal{O}_{IR}(\wp)+\mathcal{O}_{IR}(S)+\cdots} \right\rangle_{IR} \tag{5.1}$$

The reason is that we can scale up the metric: We replace:

$$g_{\mu\nu} \to t g_{\mu\nu} \tag{5.2}$$

and we take the limit $t \to +\infty$.

On the one hand, changing t is a Q-exact change in the path integral: It cannot change the integral.

On the other hand, from the physical point of view, we are stretching lengths to infinity, and correspondingly scaling energies to zero. That is, we are studying dynamics infinitesimally above the vacuum. Therefore, it must be possible to evaluate the partition function in the low energy effective theory. (By *definition* of a LEET!!)

Our goal is going to be to make (5.1) as explicit as possible.

Note:

1. We will find that N=2 SYM has many quantum vacua. Because X is *compact* we must integrate over *all* the vacua on the RHS. They mix under tunneling and it is impossible to separate them.

2. We will also need to map the operators. Under the RG $\mathcal{O}(\wp) \to \mathcal{O}_{IR}(\wp)$ and $\mathcal{O}(S) \to \mathcal{O}_{IR}(S)$. The $+\cdots$ on the RHS indicates that there will be a subtlety in this mapping related to "contact terms."

Therefore, we need to understand the vacuum structure of the theory, first on $\mathbb{R}^4$.

5.2 Spontaneous symmetry breaking

Let us focus on the $G = SU(2)$ $N = 2$ SYM with no matter hypermultiplets.

When we work on $\mathbb{R}^4$ we specify a vacuum by the behavior of the fields at infinity. There will be no tunneling between vacua because of the infinite volume of $\mathbb{R}^3$.

In the classical theory the vacuum energy is $V = \mathrm{Tr}([\phi^*, \phi])^2$. It is minimized by normal matrices and the classical vacua are parametrized by the gauge invariant parameter

$$v = \mathrm{Tr}\phi^2 \tag{5.3}$$

v can take any value in the complex plane, and conversely, a choice of v uniquely determines a classical vacuum of the theory on $\mathbb{R}^4$. At every point on the v-plane the gauge group is spontaneously broken:

$$SU(2) \to U(1) \tag{5.4}$$

except at $v = 0$. So, except at $v = 0$ the low energy theory is just $U(1)$ N=2 Maxwell theory.

FIGURE: COMPLEX v-PLANE. POINT AT ORIGIN HAS ENHANCED GAUGE SYMMETRY.

In their breakthrough work in the spring of 1994 Seiberg and Witten showed that in the *quantum* theory we can still define:

$$u := \langle \mathcal{O} \rangle \sim \langle \mathrm{Tr}\phi^2 \rangle \tag{5.5}$$

and that the set of vacua is still the entire complex u-plane, with u uniquely labeling a quantum vacuum but now, for *every* point on the u plane the gauge group is spontaneously broken $SU(2) \to U(1)$.

FIGURE: COMPLEX u-PLANE. TWO SINGULAR POINTS.

However, they also discovered a very important fact that at $u = \pm\Lambda^2$, where Λ is the dynamically generated scale in the theory, the IR description of the theory must change. We will explain how it changes below. This in general will add *new quantum vacua.*

5.3 Seiberg and Witten's Effective action on the Coulomb branch

In this section we stay away from the special points $u = \pm\Lambda^2$.

Seiberg and Witten also gave a description of the low energy effective action as a function of u.

In general, an $N = 2$ $U(1)$-gauge theory has an action which is determined by a single holomorphic function. The vm has complex scalar fields $a, \bar{a}$, and a $U(1)$ gauge field A_μ.

The Lagrangian is:

$$\begin{aligned}\mathcal{L}_{IR,vm} \sim\ & i(\bar{\tau}F^+F^+ + \tau F^-F^-) \\ & + \mathrm{Im}\tau da * d\bar{a} + \mathrm{Im}\tau D * D \\ & + \tau\psi * d\eta + \bar{\tau}\eta d * \psi + \tau\psi d\chi - \bar{\tau}\chi(d\psi) \\ & + i\frac{d\bar{\tau}}{d\bar{a}}\eta\chi(D + F^+) + \cdots\end{aligned} \tag{5.6}$$

1. Here we have given it in the topologically twisted form we need and the $+\cdots$ contain other complicated interaction terms we will not need (but they would be relevant on non-simply connected manifolds).

2. As far as the constraint of $N = 2$ supersymmetry is concerned $\tau(a)$ can be an arbitrary holomorphic function of a. To give $\tau(a)$ is to specify the Lagrangian. Therefore, to specify the low energy theory we we need to:

   a.) Compute $\tau(a)$

   b.) Explain how $\tau(a)$ is related to u.

Seiberg-Witten's solution to this problem is the following: (We will not try to justify the solution - for that see Prof. Nekrasov's lecture. We merely state the result.)

1. Consider the family of elliptic curves:

$$E_u : \qquad y^2 = (x - u)(x - \Lambda^2)(x + \Lambda^2) \tag{5.7}$$

2. For reasons which will be clear in a second we equip these curves with a meromorphic one-form

$$\lambda_{SW} := \frac{dx}{y}(x - u) \tag{5.8}$$

♣Here this is a slight lie, since Moore and Witten used the isogenous curve with modular group $\Gamma^0(4)$ rather than the modular curve for $\Gamma(2)$. But then the singularities of the family are not manifest. ♣

3. Next we *choose* a Lagrangian homology basis A, B of $H_1(E_u)$ and *define*

$$a = \oint_A \lambda_{SW}. \tag{5.9}$$

4. Then τ is the period of E_u with respect to this homology basis - this tells us the function $\tau(a)$, *and* we have -at least implicitly - explained how the (vev of) a is related to the vacuum u. The relation is given by (5.9). Note that, as is standard in discussions of LEETs, we are using the same notation, a, for a field, such as $a(x^\mu)$ on $\mathbb{R}^4$ and its vacuum expectation value on $\mathbb{R}^4$.

5. Of course, we have made an arbitrary *choice* of homology basis, but the effective theory does not depend on this choice because a change of Lagrangian homology basis corresponds to an electromagnetic duality transformation on the abelian theory. In certain regions of the u-plane there is a preferred choice: This is the choice that gives large $\mathrm{Im}\tau$, which corresponds to the weak coupling description.

This solution of the vacuum structure leads to a very notable phenomenon: The local system $H_1(E_u; \mathbb{Z})$ has monodromy around the discriminant locus $u = \pm\Lambda^2$ where the fibration $E \to \mathbb{C}$ becomes singular.

FIGURE: SHOW TORI OVER u-PLANE DEGENERATING AT 2 POINTS.

The natural homology basis at large $|u|$ (which corresponds to the classical description of the spontaneously broken theory) has the property that the dual period

$$a_d := \oint_B \lambda_{SW} \tag{5.10}$$

goes to zero for $u \to \Lambda^2$. Similarly, $a + a_d \to 0$ at $-\Lambda^2$.

In general the vacua $u = \Lambda^2$ and $u = -\Lambda^2$ are related by a symmetry so for brevity we will only describe what happens at $u = \Lambda^2$.

5.4 BPS states

There are actually *two* things we expect to be able to solve for exactly in $N = 2$ field theories. The first is the low energy effective action. This was completely solved for the $SU(2)$ theory (including couplings to matter hypermultiplets) by Seiberg and Witten [22, 21], and has been generalized to a large number of theories by many other physicists. It is still not known how to write the SW family of curves and the SW differential for an arbitrary N=2 field theory.

The second thing we expect to solve for is the "BPS spectrum." These are the *lightest* particles in a fixed charge sector (of the low energy abelian gauge theory). Again, SW found the BPS spectrum for the pure $SU(2)$ theory and there has been much progress in the meantime in understanding that spectrum for many other theories, but again the general solution has not been achieved. This fascinating topic requires a course all by itself...

The important thing for our story today is that in the pure $SU(2)$ theory we know the BPS spectrum exactly. The particles are all heavy - and need not be included in the LEET

- with the crucial exception that at $u = \pm\Lambda^2$ precisely one BPS multiplet of particles becomes massless. That means that the low energy effective field theory used above is *invalid* at $u = \pm\Lambda^2$. Indeed the Lagrangian becomes singular at this point.

5.5 The low energy theory near $u = \Lambda^2$

Although the description (5.6) breaks down at $u = \Lambda^2$ it is clear how to correct the low energy description near each point. The light BPS particle turns out to correspond to a hypermultiplet field. At $u = \Lambda^2$, if one makes an electromagnetic duality transformation to the frame in which the vectormultiplet is $(a_d, \bar{a}_d, A_{\mu,d}, \dots)$ then the monopole field M has charge 1, i.e. the representation R is the fundamental representation of $U(1)$.

One simply adds (5.6) (in the appropriate duality frame) to the standard Lagrangian for hypermultiplets $\mathcal{L}_{HM}$ (which can be looked up in textbooks on supersymmetry. See e.g. [9].)

Now, the theory

$$\mathcal{L}_{IR,VM}(a_d, \dots) + \mathcal{L}_{HM}, \tag{5.11}$$

when topologically twisted, is again in standard MQ form, but this time the Q-fixed point equations become the Seiberg-Witten equations

$$\begin{aligned} F(A_d)^+ &= (\bar{M}M)^+ \\ \gamma \cdot DM &= 0 \end{aligned} \tag{5.12}$$

So, *there is a new branch of quantum vacua*, which we will call the Higgs branch.

We must stress that the gauge field used in (5.12) is *dual* to the one used on the Coulomb branch.

Therefore, the IR evaluation of $Z^{\xi}_{DW}(p, S)$ involves a sum of two terms:

$$Z^{\xi}_{DW}(p, S) = Z_{IR,Coulomb} + Z_{IR,Higgs} \tag{5.13}$$

Of course, the term $Z_{IR,Higgs}$ is itself a sum of two path integrals, one for the contribution at $u = \Lambda^2$ and one for the contribution at $u = -\Lambda^2$.

5.6 Mapping operators from UV to IR

Now we must understand how to express the operators $\mathcal{O}$ and $\mathcal{O}(S)$ in the low energy effective theory.

The secret is to understand how the 0-operator maps under the RG and then to realize that the K operator of (4.13) is RG invariant: The supersymmetry operators do not evolve with scale.

5.6.1 Mapping operators on the Coulomb branch

The operator $\mathcal{O}$ is the same as u, by definition. This is true both at low and high energy. The expression for u in terms of fields will be very different in the UV and IR theories.

In any case, in the IR theory we obtain $\mathcal{O}^{(1)}$ by acting with K on u using the fields and supersymmetry transformation laws in the low energy effective abelian theory. Then using standard supersymmetry transformations one finds:

$$\begin{aligned} Ka &\sim \psi \\ K\psi &\sim (F^- + D) \end{aligned} \tag{5.14}$$

and so forth. (Recall that D is a self-dual 2-form.)

Thus in the low energy theory $\mathcal{O}^{(1)} = Ku \sim \frac{\partial u}{\partial a}\psi$, and acting with K again we get

$$\begin{aligned} \mathcal{O}_{IR,c} &= u \\ \mathcal{O}_{IR,c}(S) &\sim \int_S \frac{\partial u}{\partial a}(F^- + D) + \frac{\partial^2 u}{\partial a^2}\psi^2 \end{aligned} \tag{5.15}$$

5.6.2 Mapping operators on the Higgs branch

Similarly, on the Higgs branch there is only one 0-operator with the right ghost charge, and it is a_d. The operator $\mathcal{O} = u$ is a known function of a_d, expressed in terms of modular functions. Therefore, by exactly the same strategy as we used on the Coulomb branch, we find the low energy operators

$$\begin{aligned} \mathcal{O}_{IR,h} &= u \\ \mathcal{O}_{IR,h}(S) &\sim \int_S \frac{du}{da_d}F(A_d) + \frac{d^2 u}{da_d^2}\psi^2 \end{aligned} \tag{5.16}$$

where A_d is the $U(1)$ gauge field in the duality frame in which the monopole is purely electrically charged.

5.6.3 Contact terms

In evaluating correlation functions of the operators $\mathcal{O}(S)$ in the low energy effective theory there is an important subtlety. When S has self-intersections there will be singularities even in the topological field theory which must be accounted for. In the IR theory one must insert a local operator at the points of self-intersection of S.

FIGURE: TWO INTERSECTING SURFACES

If one works with off-shell susy this must be a Q-closed operator associated with a point, and hence is just a holomorphic function of u. Thus we have:

$$\langle e^{p\mathcal{O}+\mathcal{O}(S)}\rangle_{UV} = \langle e^{p\mathcal{O}_{IR,c}+\mathcal{O}_{IR,c}(S)+S^2 T_c(u)}\rangle_{\rm Coulomb} + \langle e^{p\mathcal{O}_{IR,h}+\mathcal{O}_{IR,h}(S)+S^2 T_h(u)}\rangle_{\rm Higgs} \tag{5.17}$$

The functions $T_c(u)$ and $T_h(u)$ can be determined by self-consistency arguments, as was done in [19]. A systematic theory of these contact terms was developed by Losev-Nekrasov-Shatashvili [10]. See also [11] for a simple derivation of the result:

$$T_c(u) \sim \frac{\partial^2 \mathcal{F}}{\partial \tau_0^2} \qquad \tau_0 \sim \log \Lambda. \tag{5.18}$$

For pure $SU(2)$ theory this is a certain weight zero almost modular form under the duality group $\Gamma^0(4)$:

$$T_c(u) = -\frac{1}{24}(\left(\frac{du}{da}\right)^2 E_2 - 8u) \tag{5.19}$$

We also denote $\hat{T}_c$ where we replace E_2 by the standard modular (but nonholomorphic) object $\hat{E}_2$.

5.7 Gravitational Couplings

Seiberg and Witten determined their effective action on $\mathbb{R}^4$. When coupling to a gravitational field there are exactly three new terms which must be taken into account.

5.7.1 Gravitational Couplings on the Coulomb Branch

Let us first discuss the couplings in the effective action on the Coulomb branch.

Because of topological invariance we can only have coupling to the metric via:

$$\Delta_{\text{grv}} S = \int_X e(u) \text{Tr} R \wedge R^* + p(u) \text{Tr} R \wedge R + \frac{i}{4} \int_X F \wedge w_2(X) \tag{5.20}$$

where F is the fieldstrength of the low energy $U(1)$ abelian gauge theory.

The first two terms exponentiate to functions in the integral over the u-plane:

$$E(u)^\chi P(u)^\sigma \tag{5.21}$$

Now, one can derive from consistency of the integrand on the u-plane (behavior under electromagnetic duality together with asymptotic behavior at weak coupling) that

$$E(u) = \alpha \left(\frac{du}{da}\right)^{1/2} \tag{5.22}$$

$$P(u) = \beta \Delta^{1/8} \tag{5.23}$$

1. Here Δ is the discriminant of the curve.
2. We have written E and P in a form which generalizes when we couple the $SU(2)$ vm to hypermultiplets (that will be important when we come to geography, later.) In the pure $SU(2)$ theory $\Delta = u^2 - \Lambda^2$.
3. Here α, β are numerical constants. Ultimately they will be the *only* unknowns in the full computation, and will be "fit to the experimental data." The can be provided by

   a. Explicit computations of the Donaldson polynomials in two special cases.

   b. The wall-crossing-formula.

   c. The blowup formula.

   We will use the fit to the wall-crossing-formula.

The third term is a phase which Witten deduced from a careful treatment of the fermionic measure [28]. To state this phase properly recall that in the UV we have an $SU(2)$ bundle with 't Hooft flux $\xi \in H^2(X, \mathbb{Z}/2\mathbb{Z})$. We choose an integral lift $2\lambda_0 = \bar{w}_2(E)$ of ξ. Then the line bundles which arise in the low energy abelian gauge theory have "first Chern class" in the torsor

$$\lambda \in \Gamma_\xi := \lambda_0 + \bar{H}^2(X) \tag{5.24}$$

One way to think of this is that the spontaneous symmetry breaking reduces the $SU(2)$ structure group so that the associated rank 2 bundle E is decomposed as a sum of line bundles $L \oplus L^{-1}$. However, when there is nonzero 't Hooft flux E does not exist, but its symmetric square does

$$\mathrm{Sym}^2(L \oplus L^{-1}) = L^2 \oplus \mathcal{O} \oplus L^{-2} \tag{5.25}$$

and we are writing $2\lambda = c_1(L^2)$.

In any case, in evaluating Z_{Coulomb} we need to sum over such line bundles, and the proper version of the phase from the third term in $\Delta_{\mathrm{grv}} S$ is:

$$e^{2\pi i \lambda_0^2}(-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \tag{5.26}$$

5.7.2 Gravitational Couplings on the Higgs Branch

There is a similar story for the gravitational couplings on the Higgs branch. There will be unknown functions:

$$\Delta_{\mathrm{grv}} S_{Higgs} = \int_X e_h(u)\mathrm{Tr} R \wedge R^* + p_h(u)\mathrm{Tr} R \wedge R + c(u)F^2 + \frac{i}{4}F w_2(E) \tag{5.27}$$

Where F is now the fieldstrength of the *dual photon*. Indeed, this term is deduced from electromagnetic duality. These exponential to

$$e^{2\pi i(\lambda_0^2 + \lambda\cdot\lambda_0)} C(u)^{\lambda^2} P_h(u)^\sigma E_h(u)^\chi \tag{5.28}$$

where now λ is a spin-c structure and $2\lambda_0$ is the integral lift of $w_2(E)$ we used before.

A key point is that the first three functions e_h, p_h and c are undetermined and cannot easily be found from first principles.

However, the functions C, P, E are universal - in the sense that they are independent of the 4-fold X - and this, together with the wall-crossing phenomenon of the u-plane integral will allow us to determine them.

6. General Form of the Higgs Branch Contribution

Let us sketch now how to evaluate $Z_{IR,Higgs}$ in terms of the unknown functions T_h, C, P, E.

As we said, $Z_{IR,Higgs}$ is a sum of two terms, related by a symmetry, at $u = \Lambda^2$ and $u = -\Lambda^2$.

We focus on $u = \Lambda^2$. This term can be written as:

$$Z_{IR,Higgs,\Lambda^2} = \sum_{\lambda \in \Gamma_w} \left\langle e^{pu + \frac{i}{4\pi}\int_S \frac{du}{da_d} F(A_d) + S^2 T_h(u)} \right\rangle_{u=\Lambda^2, \lambda} \tag{6.1}$$

where we sum over the first Chern class λ of spin-c structures as in (4.25).

Now we evaluate the path integral in a fixed "flux sector" λ. The low energy effective action coupled to the light "monopole hypermultiplets" is in standard MQ form for localizing on the SW equations. Therefore the path integral in a fixed flux sector is:

$$\int_{\mathcal{M}(\lambda)} e^{2\pi i(\lambda_0^2+\lambda\cdot\lambda_0)} e^{pu+i\frac{du}{da_d}S\cdot\lambda+S^2T_h(u)} C(u)^{\lambda^2} P(u)^{\sigma} E(u)^{\chi}. \tag{6.2}$$

Here $\mathcal{M}(\lambda)$ is the moduli space of solutions to the Seiberg-Witten equations based on spin-c structure λ. It is known to be *smooth, compact, orientable* and of dimension

$$\mathrm{vdim}\mathcal{M}(\lambda) = \frac{(2\lambda)^2-(2\chi+3\sigma)}{4} := 2n(\lambda) \qquad\qquad n(\lambda)\in\mathbb{Z} \tag{6.3}$$

In SW theory it is embedded in an infinite-dimensional manifold $\mathcal{A}/\mathcal{G}\times\Gamma(S^+\otimes L)$ of homotopy type $\mathbb{C}P^{\infty}$ and so inherits a class of degree two. All these assertions are proved in the textbooks [6, 23, 20].

In the topological field theory that class is - up to normalization, and again invoking the correspondence (3.8) - the field a_d of ghost number 2. The way we should interpret the integral (6.2) is that we expand

$$u = \Lambda^2 - 2i\Lambda a_d + \mathcal{O}(a_d^2) \tag{6.4}$$

This exact expansion is completely known in terms of elliptic functions. Then we define the *Seiberg-Witten invariant* to be:

$$\mathrm{SW}(\lambda) := \int_{\mathcal{M}(\lambda)} a_d^{n(\lambda)} \tag{6.5}$$

This is an integer, and thus we can express the contribution of a spin-c structure λ to $Z_{IR,Higgs,\Lambda^2}$ as

$$\mathrm{SW}(\lambda)\mathrm{Res}_{a_d=0}\frac{da_d}{a_d^{1+n(\lambda)}}\left(e^{pu+\cdots}C^{\lambda^2}P^{\sigma}E^{\chi}\right) \tag{6.6}$$

This is as far as we can go without an explicit knowledge of T_h, C, P, E.

7. The Coulomb branch contribution aka The u-plane integral

Now we will evaluate $Z_{IR,Coul}$.

We have described above all the ingredients that go into doing the u-plane integral.

An important scaling argument [19] shows that when X has $b_2^+>0$ then the result is determined by the *tree-level path integral*: We can forget about one-loop determinants and nontrivial Feynman graphs.

Thus, we are immediately left with a finite-dimensional integral

$$\int (da d\bar{a})(d\eta d\chi d\psi)(dA dD) e^{S+\Delta_{\mathrm{grv}}S} e^{pu+\mathcal{O}_{IR,c}(S)+S^2T_c(u)} \tag{7.1}$$

7.1 The integral over fermions

Let us first consider the fermionic integral:

1. The space $H^0(X;\mathbb{R})$ of η zeromodes is one-dimensional.
2. The space $H^1(X;\mathbb{R})$ of ψ zeromodes is b_1-dimensional.
3. The space $H^{2,+}(X;\mathbb{R})$ of χ zeromodes is b_2^+-dimensional.

There are three remarks to make about the fermionic integral:

1. Note that a choice of orientation is needed to define the Grassmann integral $d\eta d\psi d\chi$. Moreover, in pinning down the overall sign of the partition function Z_{DW} one must choose an integral lift of $w_2(E)$, here denoted $2\lambda_0$. A change of this choice $\lambda_0' = \lambda_0 + 2\beta$ changes the overall sign by $(-1)^{\beta\cdot w_2(X)} = (-1)^{\beta^2}$. All of this is beautifully mirrored in Donaldson theory where a choice of orientation of $H^0 \oplus H^1 \oplus H^{2,+}$ determines an orientation of the moduli space of instantons $\mathcal{M}$, and moreover a change of lift of $w_2(E)$ produces just the right change of sign [3]. It is reassuring to see these "fine structure details" mirrored in the physical approach.

2. For simplicity we will assume X is simply connected. This means we can drop the ψ-integral, and moreover we can drop the ψ-dependence in the action (5.6) and in the observables. This simplifies the equations a lot. The equations with $b_1 \neq 0$ have been worked out in [19] and [12].

3. Now, since there is a one-dimensional space of η zeromodes, and since η does not appear in any of the observables a glance at the action (5.6) shows that for the u-plane integral to be nonzero we must have $b_2^+(X) = 1$. This might seem discouraging, but we will press on. Note that integrating out η and χ then brings down a factor of
$$\frac{d\bar{\tau}}{d\bar{a}}(D+F)^+ \tag{7.2}$$
(Since $b_2^+ = 1$ we can regard $(D+F)^+$ as a scalar.)

7.2 The photon path integral

The integral over the gauge field is straightforward. As we have discussed, we sum over flux sectors labeled by $\lambda \in \Gamma_\xi$. The result is that we replace
$$F \to 4\pi\lambda \tag{7.3}$$
and get
$$\frac{e^{2\pi i\lambda_0^2}}{\sqrt{\mathrm{Im}\tau}} \sum_{\lambda\in\Gamma_\xi} e^{-i\pi\bar{\tau}\lambda_+^2 - i\pi\tau\lambda_-^2 - i\frac{du}{da}(S,\lambda_-)} (-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \left[4\pi\lambda_+ + D\right] \tag{7.4}$$

7.3 Final Expression for the u-plane integral

Finally, we just do the 1-dimensional (because $b_2^+ = 1$) Gaussian integral over the auxiliary field D. The final expression is: $Z_{IR,Coul} = Z_u^\xi(p,S)$ with
$$Z_u^\xi(p,S) = \alpha^\chi \beta^\sigma \int da d\bar{a} \frac{d\bar{\tau}}{d\bar{a}} \left(\frac{du}{da}\right)^{\chi/2} \Delta^{\sigma/8} e^{pu + S^2\hat{T}_c(u)} \Theta \tag{7.5}$$

$$\Theta = \frac{e^{-\frac{S_-^2}{8\pi y}\left(\frac{du}{da}\right)^2}}{\sqrt{y}} e^{2\pi i \lambda_0^2} \sum e^{-i\pi\bar{\tau}\lambda_+^2 - i\pi\tau\lambda_-^2 - i\frac{du}{da}(S,\lambda_-)} (-1)^{(\lambda-\lambda_0)\cdot w_2(X)} \left[\lambda_+ + \frac{i}{4\pi y}\frac{du}{da}S_+\right] \tag{7.6}$$

where we define $y = \mathrm{Im}\tau$.

Let us make a number of comments about this result for the u-plane integral

1. The expression has been written in a form valid for the inclusion of hypermultiplets (in the rank one case). That will be useful later.

2. We can rewrite the integral as

$$\int da d\bar{a}(\cdots) = \int_{\mathbb{C}} du d\bar{u}\ |\frac{da}{du}|^2\ (\cdots) \tag{7.7}$$

   However, notice that the integrand makes (extensive!) use of a duality frame. It is not at all obvious that the expression is in fact a well-defined measure on the u-plane, but this can be checked using the modular properties of the various terms. In particular, Θ is essentially a theta-function and has nice duality transformation properties.

3. In pure $SU(2)$ theory it is better to write

$$\int da d\bar{a}(\cdots) = \int_{\mathcal{F}} d\tau d\bar{\tau}\ |\frac{da}{d\tau}|^2\ (\cdots) \tag{7.8}$$

   isomorphic to the fundamental domain for $\Gamma^0(4)$ on the upper-half-plane:

   FIGURE OF FUNDAMENTAL DOMAIN FOR $\Gamma^0(4)$

   Then all the factors in the integrand can be written as modular functions of τ, although the relevant expansion in q is different near $\tau = i\infty$, $\tau = 0$ and $\tau = 2$.

4. Near the discriminant locus, and $u = \infty$, various terms in the integrand become singular. One must define the integral with care in these regions.

5. While the integral is subtle and complicated, we must stress that the topology of X only enters through the classical cohomology ring, and therefore $Z_{IR,Coul}$ is only a function of the homotopy type of X.

6. Notice that although we are discussing topological field theory the integrand certainly has nontrivial metric dependence since it explicitly uses the projection of λ to its self-dual λ_+ and anti-self-dual λ_- parts. Since we are dealing with topological field theory we might hope that the result of the integral is metric independent. We next turn to a detailed study of this question.

7.4 Metric Dependence: Wall-crossing

The formalism of topological field theory guarantees that the variation of the path integral with respect to the metric will be a total derivative in field space:

$$\frac{\delta}{\delta g_{\mu\nu}} Z = \langle T_{\mu\nu} \rangle = \langle \{Q, \Lambda_{\mu\nu}\} \rangle \tag{7.9}$$

however, in some situations that total derivative will not be zero. One example is the holomorphic anomaly of BCOV. The u-plane integral is another striking example of this.

Since $b_2^+ = 1$, the cohomology space $H^2(X;\mathbb{R})$ is a vector space with a Lorentzian metric. Once we choose an orientation of $H^{2,+}$ there is a unique self-dual class ω so that $\omega \cdot \omega = 1$, so ω lies on the "mass-shell."

FIGURE: HYPERBOLOID

The metric dependence enters the u-plane integrand entirely through the projections such as

$$\lambda = \lambda_+ \omega + \lambda_- \tag{7.10}$$

Therefore we can just consider a family $\omega(t)$ along the hyperboloid and take derivatives.

One can work out the explicit total derivative and reduce the variation of Z wrt to the metric to a boundary integral:

$$\frac{d}{dt} Z_u^\xi(p,S) = -i\alpha^\chi \beta^\sigma \sum_{u*=\pm\Lambda^2,\infty} \lim_{\epsilon\to 0} \oint_{S^1(\epsilon)} du \left(\frac{da}{du}\right)^{1-\frac{1}{2}\chi} \Delta^{\sigma/8} e^{2pu+S^2 T(u)} \Upsilon \tag{7.11}$$

where Υ is another theta function similar to Θ. Close analysis shows that this is a δ-function (except for $N_f = 4$).

The support of the δ-function is at certain walls of the form

$$W(\lambda) := \{\omega : \omega \cdot \lambda = 0\} \tag{7.12}$$

The essence of the matter is that near $u*$ the gauge coupling $\mathrm{Im}\tau \to \infty$, exponentially suppressing all terms in the theta function but one, associated with a vector λ so that $\lambda_+ \to 0$. The τ integral looks like

$$c(n) \int^\infty \frac{dy}{y^{1/2}} e^{-2\pi\lambda_+^2 y} \lambda_+ \sim c(n)\mathrm{sign}(\lambda_+) \tag{7.13}$$

where $c(n)$ are coefficients of a modular form, and only the term with $n = \lambda^2/2$ survives the integral over $\mathrm{Re}\tau$.

Physically, what happens at the $u = \infty$ walls is that the connection can become reducible and there is an abelian instanton, i.e. a connection on the line bundle L with $F^+ = 0$. This leads to an extra bosonic zeromode in the path integral leading to a δ-function divergence.

The walls are located at

$$\text{From } u = \infty \ : \qquad \lambda \in \Gamma_\xi = \frac{1}{2}\bar{w}_2(E) + \bar{H}^2(X) \tag{7.14}$$

$$\text{From } u = \pm\Lambda^2 \ : \qquad \lambda \in \Gamma_w = \frac{1}{2}\bar{w}_2(X) + \bar{H}^2(X) \tag{7.15}$$

The discontinuity across the walls $\Delta_{u_*,\lambda} Z_u^\xi(p,S)$ can be expressed as a residue of a *holomorphic* object: This is the Fourier coefficient of a modular form.

The walls divide up the forward light-cone into chambers. A correlator $\langle \mathcal{O}^\ell \mathcal{O}(S)^r \rangle$ for fixed ℓ, r will only change across a finite number of chambers.

FIGURE: CHAMBERS

The metric dependence of any correlator is then piecewise constant. The *wall-crossing formula* across the walls will involve Fourier coefficients of modular forms.

The WCF for the walls coming from $u = \infty$, $\Delta_{\infty,\lambda} Z_u^\xi(p,S)$ reproduce *precisely* the formula of L. Göttsche for the change of the Donaldson polynomials for $b_2^+ = 1$ if we set:

$$\alpha^\chi \beta^\sigma = \frac{2^{(2+3\sigma)/4}}{\pi} \tag{7.16}$$

Now $\chi + \sigma = 4$, but $\sigma = 1 - b_2^-$ can vary, so this completely fixes α, β. (We have also scaled $\Lambda = 1$ in these equations.)

However, we also have a WCF across the walls coming from singularities at $u = \pm\Lambda^2$. Since we have already completely accounted for the change of the Donaldson polynomials from $\Delta_{\infty,\lambda} Z_u^\xi(p,S)$, these new discontinuities must *not* be discontinuities of the full partition function $Z_{DW}^\xi(p,S)$.

8. Derivation of the relation between SW and Donaldson invariants.

Let us recap the situation:

We have (always!)

$$Z_{DW}^\xi(p,S) = Z_u^\xi(p,S) + Z_{IR,Higgs} \tag{8.1}$$

When X has $b_2^+ = 1$ we know that $Z_u^\xi(p,S)$ has discontinuities as a function of $\omega \in H^2(X;\mathbb{R})$ across walls $W(\lambda)$ coming from the singularities $u = \infty$ and $u = \pm\Lambda^2$.

Moreover, $Z_{DW}^\xi(p,S)$ also has discontinuities, and these are perfectly accounted for by the discontinuities of $Z_u^\xi(p,S)$ coming from $u = \infty$.

Therefore, across all walls $W(\lambda)$ we must have

$$0 = \Delta_{u=\Lambda^2,\lambda} Z_{IR,Coul} + \Delta_{u=\Lambda^2,\lambda} Z_{IR,Higgs} \tag{8.2}$$

Indeed, mathematically, the SW invariant $\mathrm{SW}(\lambda)$ is known *not* to be an invariant when X has $b_2^+ = 1$ and changes across walls $W(\lambda)$ determined by spin-c structures. The WCF is particularly easy:

$$\mathrm{SW}(\lambda)|_{\omega\cdot\lambda=0^+} - \mathrm{SW}(\lambda)|_{\omega\cdot\lambda=0^-} = (-1)^{1+n(\lambda)} \tag{8.3}$$

(when crossing in a suitable direction).

Mathematically, at such walls there is a solution of the SW equations with $M = 0$, this is a reducible solution fixed under global $U(1)$ gauge transformations and the moduli space becomes singular. Physically, at these walls since $M = 0$ the Higgs and Coulomb branches can "mix."

Now, we can *compute* $\Delta_{u=\Lambda^2,\lambda} Z_{IR,Coul}$ since we have an explicit expression (7.11) for it and, given the general form (6.6) of the Higgs contribution and the SW WCF (8.3) we can *compute* the unknown couplings $C(u), P(u), E(u)$. For example, we find

$$\begin{aligned} C(u) &= \left(\frac{a_d}{q_d}\right)^{1/2} = 4e^{i\pi/4} + \mathcal{O}(a_d) \\ P(u) &= e^{i\pi/32} 2^{5/4} + \mathcal{O}(a_d) \\ E(u) &= e^{i\pi/8} 2^{3/4} + \mathcal{O}(a_d) \end{aligned} \tag{8.4}$$

where $q_d = e^{2\pi i \tau_d}$. These are completely explicitly known series determined by modular functions.

To summarize: we now have a completely explicit expression for $Z^{\xi}_{DW}(p,S)$, expressed in terms of the SW invariants and the classical cohomology ring. It is valid for all simply connected 4-folds with $b_2^+ > 0$, and can be easily generalized to include the non-simply-connected case.

9. Simple Type and Witten's Conjecture

A key property about the Seiberg-Witten invariants on a 4-fold X is that $\mathcal{M}(\lambda)$ is only nonempty for a *finite* set of λ. These are called the *basic classes.*

Now, let us define X to be of *Seiberg-Witten simple type* if $\mathcal{M}(\lambda) \neq \emptyset$ only for λ such that $n(\lambda) = 0$. In this case $\mathcal{M}(\lambda)$ is a finite union of oriented points. When evaluating $\mathrm{SW}(\lambda)$ we are literally counting solutions to equations, just as we began our lecture.

It is a strange fact that all known simply connected X with $b_2^+ > 1$ are of Seiberg-Witten simple type, but there is no proof that all such X must be of simple type.

In any case, let us now suppose that X is of SW simple type, and moreover that $b_2^+ > 1$. In that case $Z_{IR,Coul} = 0$ and the integral is given entirely by the contributions at $u = \pm\Lambda^2$. Moreover, these are easily evaluated since $n(\lambda) = 0$. Putting it all together we obtain the key statement of [27], referred to as the "Witten conjecture" in the mathematics literature:

$$\begin{aligned} Z^{\xi}_{DW}(p,S) = 2^{1+\frac{1}{4}(7\chi+11\sigma)} \Bigg(& e^{\frac{1}{2}S^2+p} \sum_{\lambda\in\Gamma_w} \mathrm{SW}(\lambda) e^{2\pi i(\lambda\cdot\lambda_0+\lambda_0^2)} e^{2S\cdot\lambda} \\ & + i^{\chi_h} e^{-\frac{1}{2}S^2-p} \sum_{\lambda\in\Gamma_w} \mathrm{SW}(\lambda) e^{2\pi i(\lambda\cdot\lambda_0+\lambda_0^2)} e^{-i2S\cdot\lambda} \Bigg) \end{aligned} \tag{9.1}$$

where $\chi_h := (\chi+\sigma)/4$.

Here we have set $\Lambda = 1$. The first sum comes from the monopole point, $u = \Lambda^2$ and the second sum comes from $u = -\Lambda^2$, the dyon point.

Now, in their analysis of Donaldson polynomials Kronheimer and Mrowka introduced the idea of *simple type* - which we will call KM simple type. It says that the partition function Z_{DW} satisfies the simple differential equation:

$$\left(\frac{\partial^2}{\partial p^2}-1\right) Z^{\xi}_{DW}(p,S)=0 \tag{9.2}$$

We note that from our general physical expression it is an immediate consequence that for $b_2^+>1$, if X is of SW simple type then it is of KM simple type.

KM also introduced a notion of *generalized simple type.* This says that for some r

$$\left(\frac{\partial^2}{\partial p^2}-1\right)^r Z^{\xi}_{DW}(p,S)=0 \tag{9.3}$$

Note that we have a physical proof that *all* simply connected 4-folds of $b_2^+>1$ are of generalized KM simple type. This is a simple consequence of the fact that there are only a finite number of basic classes. Therefore, we can take

$$r = 1+\max_{\lambda} n(\lambda) \tag{9.4}$$

Remark There is a beautiful interpretation of the localization of Z_{DW} in terms of localization to $N=1$ vacua. The essential idea is that sometimes (e.g. on a Kahler manifold) one can add a mass term for the fermions breaking $N=2$ to $N=1$, but preserving a topological symmetry. See [26]

10. Applications of the u-plane integral.

The SW equations had immediate mathematical applications. Having spent years building up an arsenal of techniques for dealing with the much more difficult nonabelian equations all of Donaldson's theorems were reproven with the SW equations in a matter of months. Moreover the SW equations led to the resolution of long-standing conjectures (such as the Thom conjecture), and allowed mathematicians to go beyond what had been achieved with Donaldson's theory.

For a lucid account see the review by S. Donaldson [4]. The account is absolutely masterful, except for one paragraph where he tries to explain the reasoning behind Witten's conjecture. Here Donaldson just goes to pieces.

By and large, having gotten the hint that one could work with the Seiberg-Witten equations instead of the nonabelian anti-self-dual equations the mathematicians have *not* really used the physical insights I have just explained. One exception could have been the work by Taubes. In some beautiful work he shows that the SW invariants on a *symplectic manifold* can be identified with Gromov-Witten invariants counting pseudoholomorphic curves. This could have been predicted by physicists from the physics of superconductivity, since the SW equations are very similar to the equations for the Landau-Ginzburg low energy effective theory of superconductivity. The pseudoholomorphic curves in question can be thought of physically has worldsheets of Abrikosov-Gorkov flux lines. See the beautiful article by Witten [29] for an account of this. Unfortunately, for the physicists, this was a physics *postdiction.*

Nevertheless, the physical approach we have outlined does have a number of less well-known applications which we would like to advertise:

1. It gives a simple physical derivation of the Fintushel-Stern /Gottsche-Zagier blowup formula as a kind of "operator product expansion" of the 2-observable for the exceptional surface of a blowup. Topologically a blowup is $\hat{X} = X\#\overline{\mathbb{C}P^2}$ and one can show easily from the u-plane that

$$\exp[t\mathcal{O}(E)] = \sum_{k=1}^{\infty} t^k \mathcal{B}_k(\mathcal{O}) \tag{10.1}$$

where $\mathcal{B}_k(\mathcal{O})$ are polynomials. This is a 4d version of the familiar maneuver in 2d CFT of replacing a handle by an infinite sum over local operators. Details are in [19].

2. It gives explicit formulae for Z_{DW} by actually doing the integral in some good cases. Most notably, the answer for $\mathbb{C}P^2$ highlights an intriguing relation to class numbers of quadratic imaginary fields and Mock modular forms. Indeed the u-plane integral is closely related to certain kinds of "Θ-lifts" which have appeared in number theory as well as in string perturbation theory. In the latter context they have been used to give conceptual proofs of Borcherds' results on automorphic products. [Cite: HarveyMoore].

3. It leads to nontrivial relations to 3-manifold topology, the Casson-Walker-Lescop invariant, and Reidemeister-Milnor torsion. Some of these relations raised puzzles which have never been fully resolved. See [14].

4. The technique sketched above has a relatively straightforward generalization to higher rank invariants. There is an analog of the above formulae for the $SU(N)$ Donaldson invariants [13]. Dissapointingly, it is again completely expressed in terms of the classical cohomology ring and the Seiberg-Witten invariants. Kronhiemer has verified that prediction for some special X's [8].

5. The relation of the topology of 4-folds to the existence of superconformal fixed points led to some nontrivial new results in topology [15, 16].

Time precludes a discussion of all these applications, but we would like to say a little bit about the application to geography and "superconformal simple type."

10.1 The geography problem

To a compact 4-fold we can associate $(\chi, \sigma, t) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ where t is the type, telling us whether the intersection form is even or odd.

The geography problem asks which values can occur, and for a given (χ, σ, t) how many examples (i.e. nondiffeomorphic manifolds) are there? For an excellent summary see [24, 5].

Regarding the uniqueness, it is clear we need to put some restrictions to avoid trivialities. For example,

$$n\mathbb{C}P^2 \# m\overline{\mathbb{C}P^2} \tag{10.2}$$

has $\chi = 2 + m + n$ and $\sigma = n - m$, and since $\chi + \sigma = 2(1 + b_2^+ - b_1)$ is even there are essentially no restrictions on χ, σ, except for those from $m \geq 0, n \geq 0$.

Thus we can look at:

1. Complex manifolds.
2. Symplectic manifolds.
3. Irreducible manifolds (This means $X = X_1 \# X_2$ implies X_1 or X_2 is S^4).

It is best to plot the bounds in terms of

$$c := 2\chi + 3\sigma \qquad \chi_h := \frac{\chi + \sigma}{4} = \frac{1 + b_2^+ - b_1}{2} \tag{10.3}$$

χ_h can be integer or half-integer. If it is integer X admits an almost complex structure. Then $c = c_1(X)^2$. If X is complex χ_h is the holomorphic Euler characteristic.

FIGURE1: SOME KNOWN BOUNDS. Plot minimal surfaces of general type: $2\chi_h - 6 \leq c \leq 9\chi_h$.

10.2 Superconformal singularities

Now it turns out that the physics of *superconformal points* actually has some bearing on the geography problem.

The way this comes about is the following. As we have stressed, Witten's formula has a natural generalization to $SU(2)$ SYM coupled to N_f hypermultiplets with R the fundamental representation. The UV quantum field theory is only well-defined for $N_f \leq 4$, so we restrict to this case. Each hypermultiplet comes with a complex "mass parameter" m_i. (Mathematically, the m_i are parameters in equivariant cohomology, a result of Labastida and Marino [cite]).

Once again, the quantum moduli space of vacua is the complex plane, parametrized by $u \in \mathbb{C}$, but now the curves Σ_u in the Seiberg-Witten family over the u-plane degenerate at $2 + N_f$ points u_j, $j = 1, \ldots, 2 + N_f$. At each of these points a different kind of BPS state becomes massless:

FIGURE OF U-PLANE WITH SEVERAL SINGULARITIES

For X with $b_2^+ > 1$ of SW simple type the partition function becomes:

$$Z_{DW}(p, S; m_i) = \tilde{\alpha}^\chi \tilde{\beta}^\sigma \sum_{j=1,\ldots,2+N_f} \kappa_j^{\chi_h} \left(\frac{du}{da}\right)_j^{\chi_h + \sigma} \sum_\lambda \mathrm{SW}(\lambda) e^{pu_j + S^2 T_j - i\left(\frac{du}{da}\right)_j S\cdot\lambda} \tag{10.4}$$

Here

1. $\tilde{\alpha}$ and $\tilde{\beta}$ are slightly different numerical constants from before. We have put $\Lambda_{N_f} = 1$.
2. κ_j is defined by $u = u_j + \kappa_j q_j + \cdots$, where $q_j = e^{2\pi i \tau_j}$ is the relevant modular parameter near the singularity u_j.

Now, $Z_{DW}(p, S; m_i)$ is a manifestly finite and well-defined expression for *generic* values of the mass parameters m_i. However, as we vary the mass parameters the points in the discriminant locus u_j will move, and they can even collide. When that collision involves massless particles which are both magnetically and electrically charged [1] there are further singularities. Mathematically, this is familiar in Kodaira's classification of elliptic fibrations.

Now let us focus on $N_f = 1$. For $N_f = 1$ there is a point m_* where two singularities collide at a single point u_*. If we parametrize $m = m_* + z$ and $u = u_* + z + \delta u$ then the Seiberg-Witten curve is, to leading order:

$$y^2 = x^3 + zx + \delta u \tag{10.5}$$

up to numerical coefficients. There are then extra zeroes in κ_j and $\left(\frac{du}{da}\right)_j$. Since $\chi + \sigma$ might well be negative there are potential divergences in $Z_{DW}(p, S; m)$ as $m \to m_*$.

However, from the physical perspective there cannot be any such divergences when X is a compact manifold.

The reason is that in the IR the only singularities can come from noncompact regions in spacetime or in moduli space. But X is compact, and for $N_f = 1$ there are no such noncompact regions. (For $N_f > 1$ superconformal singularities sometimes can involve noncompact regions.)

Requiring that $Z_{DW}(p, S; m)$ as $m \to m_*$ turns out to imply nontrivial facts about the SW invariants.

10.3 Superconformal simple type and the generalized Noether inequality

A close analysis of the potential singularities of $Z_{DW}(p, S; m)$ shows that the absence of a divergence for $m \to m_*$ is guaranteed by the following mathematical criterion:

Define

$$\mathrm{SW}_X(z) := \sum_\lambda e^{2\pi i \lambda_c \cdot \lambda} \mathrm{SW}(\lambda) e^{z\lambda} \tag{10.6}$$

where we fix an integral lift $2\lambda_c$ of $w_2(X)$ and we regard powers λ^n to be in the dual space of $\mathrm{Sym}^n(\bar{H}^2(X))$. Then

If $\mathrm{SW}_X(z)$ is analytic at $z = 0$ with an of order $\geq \chi_h - c - 3$ then $Z_{DW}(p, S; m)$ is finite for $m \to m_*$.

We *define* X to be of *superconformal simple type* if $\mathrm{SW}_X(z)$ has a (nonnegative order) zero at $z = 0$ of order $\geq \chi_h - c - 3$. Reference [15] did not quite manage to prove that this is a *necessary* condition that $Z_{DW}(p, S; m)$ be finite, but it was verified that all available constructions of 4-manifolds satisfy this criterion. Recently [7] have proven that projective varieties are SST.

Pursuing this a little further leads to an interesting lower bound on the number of basic classes. Let B the the number of basic classes (where we count two nonzero classes λ and $-\lambda$ as the same. Then

$$B \geq \left[\frac{\chi_h - c}{2}\right] \tag{10.7}$$

[1] technically, non-mutually-local

which implies

$$c \geq \chi_h - 2B - 1 \tag{10.8}$$

A classic result of algebraic geometry is that minimal surfaces of general type satisfy

$$c \geq 2\chi_h - 6 \tag{10.9}$$

This is known as the Noether bound, so we refer to (10.8) as the "generalized Noether bound."

This leads to some new lines in the geography problem:

FIGURE: c, χ_h PLANE WITH SOME LINES WHERE THE SW SUM RULES APPLY.

11. Possible Future directions.

1. There are interesting cases, such as $S^3 \times S^1$, where one-loop terms will contribute. However, as shown in [19], the series stops at one-loop. Recently, beautiful results on the partition function of $N = 2$ theories on $S^3 \times S^1$ have been obtained by Rastelli et. al. It would be interesting to reproduce those using the u-plane integral.

2. Families of 4-manifolds and $H^*(BDiff)$. Recent progress [CITE:SEIBERG et. al.] on coupling rigid SUSY theories to background supergravity should help.

3. Give expression for the "u-plane integral" for theories of class S. What is the UV equation whose intersection theory we are computing? Can we use the vast new array of superconformal theories to learn new things, perhaps along the lines of the superconformal simple type story?

4. When $\chi_h = \frac{1-b_1+b_2^+}{2}$ is half-integral all the SW invariants vanish. Can physics really be blind to half the world of four-manifolds?

References

[1] M. F. Atiyah and L. Jeffrey, "Topological Lagrangians and cohomology," J. Geom. Phys. **7**, 119 (1990).

[2] S. Cordes, G. W. Moore and S. Ramgoolam, "Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories," Nucl. Phys. Proc. Suppl. **41**, 184 (1995) [hep-th/9411210].

[3] S. Donaldson and P. Kronheimer, *The Geometry of Four-Manifolds*, Oxford 1990.

[4] S. Donaldson, Bull. Amer. Math. Soc. Vol. 33 (1996) p. 45

[5] Six lectures on four manifolds

[6] Gompf and Stipsicz

[7] L. Göttsche, H. Nakajima, and K. Yoshioka, "Donaldson=Seiberg-Witten From Mochizuki's Formula and Instanton Counting," arXiv:1001.5024.

[8] P. Kronheimer, "Four-manifold invariants from higher-rank bundles," arXiv.math/0407518

[9] J. Labastida and M. Marino, "Topological quantum field theory and four manifolds," (Mathematical physics studies. 25)

[10] A. Losev, N. Nekrasov and S. L. Shatashvili, "Issues in topological gauge theory," Nucl. Phys. B **534**, 549 (1998) [hep-th/9711108].

[11] M. Marino and G. W. Moore, "Integrating over the Coulomb branch in N=2 gauge theory," Nucl. Phys. Proc. Suppl. **68**, 336 (1998) [hep-th/9712062].

[12] M. Marino and G. W. Moore, "Donaldson invariants for nonsimply connected manifolds," Commun. Math. Phys. **203**, 249 (1999) [hep-th/9804104].

[13] M. Marino and G. W. Moore, "The Donaldson-Witten function for gauge groups of rank larger than one," Commun. Math. Phys. **199**, 25 (1998) [hep-th/9802185].

[14] M. Marino and G. W. Moore, "Three manifold topology and the Donaldson-Witten partition function," Nucl. Phys. B **547**, 569 (1999) [hep-th/9811214].

[15] M. Marino, G. W. Moore and G. Peradze, "Superconformal invariance and the geography of four manifolds," Commun. Math. Phys. **205**, 691 (1999) [hep-th/9812055].

[16] M. Marino, G. W. Moore and G. Peradze, "Four manifold geography and superconformal symmetry," math/9812042.

[17] J. Milnor, "On simply connected 4-manifolds." 1958 Symposium internacional de topologa algebraica International symposium on algebraic topology pp. 122128 Universidad Nacional Autnoma de Mxico and UNESCO, Mexico City

[18] G. W. Moore, "2-D Yang-Mills theory and topological field theory," hep-th/9409044.

[19] G. W. Moore and E. Witten, "Integration over the u plane in Donaldson theory," Adv. Theor. Math. Phys. **1**, 298 (1998) [hep-th/9709193].

[20] J.W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Princeton University Press Mathematical Notes, 1996.

[21] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [hep-th/9407087].

[22] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431**, 484 (1994) [hep-th/9408099].

[23] A. Scorpan, *The Wild World of Four-Manifolds*, AMS,...

[24] Will we every classify 4-folds?

[25] E. Witten, "Topological Quantum Field Theory," Commun. Math. Phys. **117**, 353 (1988).

[26] E. Witten, "Supersymmetric Yang-Mills theory on a four manifold," J. Math. Phys. **35**, 5101 (1994) [hep-th/9403195].

[27] E. Witten, "Monopoles and four manifolds," Math. Res. Lett. **1**, 769 (1994) [hep-th/9411102].

[28] E. Witten, "On S duality in Abelian gauge theory," Selecta Math. **1**, 383 (1995) [hep-th/9505186].

[29] E. Witten, "From Superconductors and Four-Manifolds to Weak Interactions," Bull. Amer. Math. Soc. Vol. 44 (2007) 361.