2d Categorical Wall-Crossing With Twisted Masses, And An Application To Knot Invariants

Gregory Moore   Rutgers

Part of this talk is review of old things

Witten, 1982; Cecotti & Vafa, 1992

Some foundational material: D. Gaiotto, G. Moore, & E. Witten 2015 (GMW)

New things are work with

AHSAN KHAN

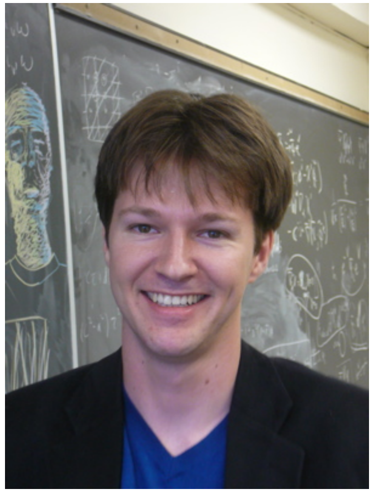
arXiv:2010.11837

arXiv:21??.????

+ comment resulting from discussions with



With nontrivial input from Andy and Tudor




1. **Supersymmetric Quantum Mechanics And Homological algebra**
2. 2D N=(2,2) Landau-Ginzburg Models
3. Thimble Branes & Their Local Operators
4. Categorical Wall-Crossing
5. Generalization To Twisted Masses
6. Relation To 3d Indices

SQM & Morse-Novikov Theory (Witten: 1982)

M: Riemannian; ξ: Closed 1-form

Locally $\xi = dh$ with $h: M \to \mathbb{R}$ the traditional superpotential.

But h need not be single-valued: ξ need not be exact.

Call ξ the ``super-one-form''

SQM: $\phi: \mathbb{R}_t \to M$

$$L = g_{IJ}(\phi)\dot{\phi}^I\dot{\phi}^J - \| \xi \|^2 + \cdots$$

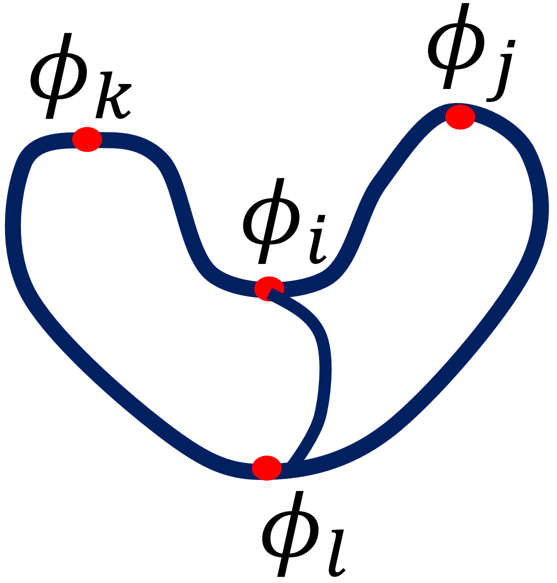
SQM & Morse-Novikov Theory (Witten: 1982)

SQM: $\phi: \mathbb{R}_t \to M$

$$L = g_{IJ}(\phi)\dot{\phi}^I\dot{\phi}^J - \| \xi \|^2 + \cdots$$

Classical vacua: $\xi(\phi_i) = 0$ ``critical points''

``Mass matrix'' $D_I \xi_J |_{\phi_i}$ is invertible



→ *Approximate* quantum vacua: $\Psi(\phi_i)$

Fermion number: $F\big(\Psi(\phi_i)\big) = \frac{1}{2}\left(d_-(\phi_i) - d_+(\phi_i)\right)$

Instantons & MSW Complex

The approximate vacua are not exact because of instanton effects.

Instanton equation: $$\frac{d\phi^I}{d\tau} = g^{IJ}(\phi)\xi_J(\phi)$$

Use instantons to define an operator Q
on approximate ground states

$$Q\Psi(\phi_i) := \sum_{F_j = F_i + 1} n_{j,i}\,\Psi(\phi_j)$$

Broken flows: $$\sum_{F_r = F_q + 1 = F_p + 2} n_{pq} n_{qr} = 0$$

$$\Rightarrow \quad Q^2 = 0$$

MSW Chain Complex

$$\mathcal{C} \coloneqq (V, F, Q)$$

SQM: $V \subset \mathcal{H}$: The span of the approximate ground states $\Psi(\phi_i)$

F: Fermion number = Homological degree

Q: Susy operator= Differential

$$[F, Q] = Q$$

$$Q^2 = 0$$

MSW complex: MSW(M, g_{IJ}, ξ_I)

Exact ground states $\cong H^*(V, Q) \cong H^*(M; d + \xi)$

Homotopies Of Metric And Super-one-form

Now consider a continuous family: $\left(g_{IJ}(\phi; s), \xi_I(\phi; s) \right)$ $s_1 \leq s \leq s_2$

How does the MSW complex change?

Define $\mathcal{U}: MSW(s_1) \rightarrow MSW(s_2)$

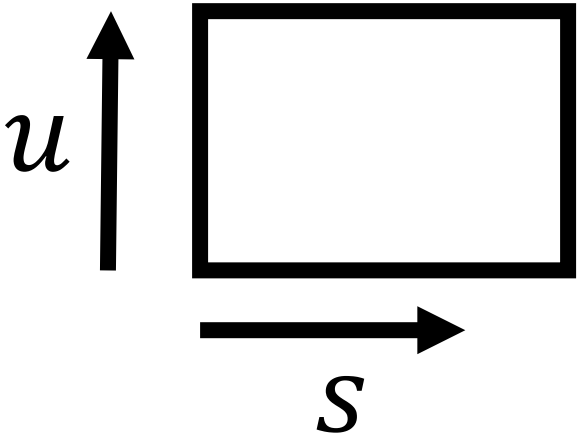
$$\mathcal{U}\,\Psi(p; s_1) = \sum_{F_q = F_p} n_{p,q}\,\Psi(q; s_2) \qquad \frac{d\phi^I}{ds} = g^{IJ}(\phi; s)\xi_J(\phi; s)$$

Claim: $\mathcal{U}\,F = F\,\mathcal{U}$ $\qquad \mathcal{U}\,Q_1 = Q_2\,\mathcal{U}$

Under continuous deformation of metric and super-one-form the MSW complex changes by a chain map.

Actually, it is a very special kind of chain map: A homotopy equivalence of chain complexes.

Homotopies Of Paths ⇒ Homotopy Of Chain Maps



$$\longrightarrow \quad Met(M) \times \{\xi' s\}$$

$$(g_{IJ}(\phi; s, u), \xi_I(\phi; s, u))$$

(Fixed @ s=0,1)

$u = 0$ gives a chain map $\mathcal{U}_0 : (V, F, Q)_1 \to (V, F, Q)_2$

$u = 1$ gives a chain map $\mathcal{U}_1 : (V, F, Q)_1 \to (V, F, Q)_2$

$$\mathcal{U}_0 - \mathcal{U}_1 = Q_2 \, E + E \, Q_1$$

$$\frac{d\phi^I}{ds} = g^{IJ}(\phi; s, u)\xi_J(\phi; s, u)$$

$$E \, \Psi(p; s = 0, u = 0) = \sum_{F_q = F_p - 1} n_{p,q} \Psi(q; s = 1, u = 1)$$

Definition: Homotopies Of Chain Maps

Two chain maps $f_0, f_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ are ***homotopic*** if

There is a Fermion number -1 map $E : V_1 \rightarrow V_2$

$$f_0 - f_1 = Q_2 E + E Q_1$$

If there are chain maps.... $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ $\quad g : \mathcal{C}_2 \rightarrow \mathcal{C}_1$

$$g \circ f \sim Id_{\mathcal{C}_1} \qquad f \circ g \sim Id_{\mathcal{C}_2}$$

Then the chain complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ are ***homotopy equivalent.***

1. Supersymmetric Quantum Mechanics And Homological algebra
2. 2D N=(2,2) Landau-Ginzburg Models
3. Thimble Branes & Their Local Operators
4. Categorical Wall-Crossing
5. Generalization To Twisted Masses
6. Relation To 3d Indices

Landau-Ginzburg Models $LG(X, \alpha)$

$X, g_{I\bar{J}}$: Kähler $\qquad \alpha \in \Omega^{1,0}(X)\ , \bar{\partial}\alpha = 0$

Locally $\alpha = \partial W$ with $W: X \to \mathbb{C}$, but we will be considering multi-valued W

$$S = \int_{\mathbb{R}} dt \int_{D} dx \ \left(g_{I\bar{J}}(\phi(x,t))\, \partial_\mu \phi^I \,\partial^\mu \phi^{\bar{J}} - \| \alpha(\phi) \|^2 + \cdots \right)$$

Poincare invariant vacua for $D = \mathbb{R}$:

$$\mathbb{V} = \{ \phi_i \mid \alpha(\phi_i) = 0 \}$$

Branes $\qquad D = [x_0, \infty)$

2d LG Model As 1d SQM

Consider SQM with target: $\mathcal{M} = Map(\phi: D \to X)$

$$\parallel \delta\phi \parallel^2 = \int_D g_{I\bar{J}}(\phi(x))\, \delta\phi^I \;\delta\phi^{\bar{J}}$$

$\alpha \in \Omega^{1,0}(X)$ induces a super-one-form ξ on $\mathcal{M}$

$$\xi[\phi] = \int_D [\phi^*(\omega) - Re(\zeta^{-1}\alpha_I(\phi)\delta\phi^I)dx\,]$$

$$\mathrm{SQM}(\mathcal{M}, \xi) = \mathrm{LG}(X, \alpha)$$

Superpotential With Twisted Masses

Usual discussion: $\alpha = \partial W$

with $W: X \to \mathbb{C}$ holomorphic and Morse

If α has nonzero periods there is no single-valued superpotential

``twisted masses''

$\exists$ Minimal Abelian cover $\pi: \hat{X} \to X$ so that $\pi^*(\alpha) = \partial\, \widehat{W}$

Γ: Free Abelian Deck group $\subset H_1(X;\mathbb{Z})$

It is often convenient to consider

$$LG(\hat{X}, \hat{\alpha} = \partial \widehat{W})$$

and work equivariantly wrt Γ

Vacua of $LG(\hat{X}, \hat{\alpha} = \partial \widehat{W})$: $\hat{\mathbb{V}} = \{ \hat{\phi}_a \mid d\widehat{W}(\hat{\phi}_a) = 0 \}$

Abbreviate vacua $\hat{\phi}_a, \hat{\phi}_b, \ldots$ simply by $a, b, \ldots$

Write free Γ $-$action on $\hat{\mathbb{V}}$: $a \to a + \gamma$

$$\widehat{W}_{a+\gamma} = \widehat{W}_a + \oint_\gamma \alpha$$

Example 1: Mirror Of The Free Chiral

$$X = \mathbb{C}^* \qquad \alpha = \left(\frac{m}{\phi} - 1\right) d\,\phi \qquad \mathbb{V} = \{\, \phi_0 = m \neq 0 \,\}$$

$$\pi: \hat{X} = \mathbb{C} \;\to\; X = \mathbb{C}^* \qquad \pi: \hat{\phi} \to \phi = \exp(\hat{\phi}) \qquad \Gamma \cong \mathbb{Z}$$

$$\hat{\alpha} = d\,\widehat{W} = d\left(m\,\hat{\phi} - e^{\hat{\phi}}\right) \;\leftarrow\; \alpha = \left(\frac{m}{\phi} - 1\right) d\,\phi$$

$$\widehat{\mathbb{V}} = \{\, \hat{\phi}_a = \log m + 2\pi i\, a \mid a \in \mathbb{Z}\}$$

$$\widehat{W}_a = m \log m + 2\pi i\, a\, m \qquad \widehat{W}_{a+n} = \widehat{W}_a + 2\pi i\, m\, n$$

Twisted mass

Other Examples

Mirror of $\mathbb{CP}^1$: $\alpha = \left(\frac{t}{\phi^2} + \frac{m}{\phi} + t\right) d\,\phi \qquad \phi \in X = \mathbb{C}^*$

Discussed in GMW framework in Galakhov (2021) and Khan-Moore, to appear

There are two vacua ϕ_i, ϕ_j and rank one deck group $\Gamma \cong \mathbb{Z}$

LG models for knot homology
[Gaiotto-Witten; Galakhov-Moore; Aganagic]

Chern-Simons-Landau-Ginzburg

$G_{\mathbb{C}}$: Complex Lie group

M_3: Riemannian 3-fold $\Rightarrow$ LG model $\mathrm{CSLG}[G_{\mathbb{C}}, M_3]$

$$X = \{\, Complex\ G_{\mathbb{C}} - connections\ on\ a\ 3 - manifold\ M_3 \,\}$$

$$\alpha = \int_{M_3} Tr\, \mathcal{F}^2 \qquad \text{``} = d\, CS(\mathcal{A}) \text{''}$$

Vacua on $D = \mathbb{R}$: Flat $G_{\mathbb{C}}$ $-$ connections

Morse Theory Flows In LG Language

$SQM(\mathcal{M},\xi)$ vacua:

$$\xi = 0 \qquad \Leftrightarrow \qquad \boxed{\frac{\partial \phi^I}{\partial x} = i\,\zeta\, g^{I\bar{J}}\bar{\alpha}_{\bar{J}}(\phi)}$$

We call this the ζ-*soliton* equation

$$\frac{\partial \phi}{\partial \tau} = \xi \qquad \Leftrightarrow \qquad \boxed{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \tau}\right)\phi^I = i\,\zeta\, g^{I\bar{J}}\,\bar{\alpha}_{\bar{J}}(\phi)}$$

We call this the ζ-*instanton* equation

Soliton Complexes For $SQM(\mathcal{M},\xi)$ & $D = \mathbb{R}$

$\phi(x) \to \phi_i$ $\phi(x) \to \phi_j$

$$R_{ij} = \left(Span\{\Psi[\phi_{ij}(x)]\}, F_{ij}, Q_{ij} \right)$$

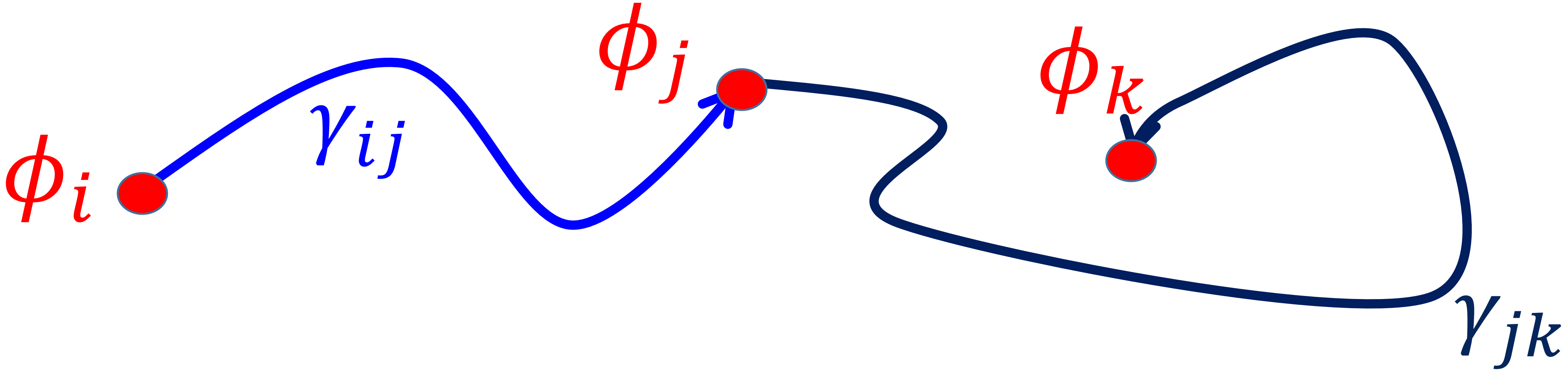
$F_{ij}\,\Psi_{ij} = \eta\left(D_{\phi_{ij}}\right)\Psi_{ij}$ Q_{ij} : Count ζ −instantons

``Flavor Charge'' : $[\phi_{ij}(\mathbb{R})] = \gamma_{ij} \in \Gamma_{ij}$

Γ_{ij} = paths in X from ϕ_i to ϕ_j --- up to homology.

Adding Charges

Composition of curves defines $\gamma_{ij} + \gamma_{jk} \in \Gamma_{ik}$



Abelian group structure on $\Gamma_{ii} \cong \Gamma$.

Γ_{ij} is a Γ −torsor

R_{ij} is graded by Γ −torsor Γ_{ij} $\quad R_{ij} = \oplus_{\gamma_{ij}\in\Gamma_{ij}} R_{\gamma_{ij}}$

``BPS index'' $\quad \mu_{\gamma_{ij}} := Tr_{R_{\gamma_{ij}}} e^{i\,\pi\, F_{ij}}$

Central charge: $Z_{\gamma_{ij}} = \oint_{\gamma_{ij}} \alpha$

Generalizes the standard (α exact) $Z_{ij} = W_i - W_j$

``Twisted mass property'' $\quad Z_{\gamma_{ij}+\gamma} = Z_{\gamma_{ij}} + Z_{\gamma}$

Periodic Solitons

Qualitatively new feature with twisted masses:

$\phi(x) \to \phi_i$ $\phi(x) \to \phi_i$

When α is not exact there can be nontrivial solutions!

$$Z_{\gamma_{ii}} = \oint_{\gamma} \alpha = \oint_{\phi_{ii}(\mathbb{R})} \alpha$$

R_{ii} can be nontrivial!

Main new ingredient in categorified wall-crossing with twisted masses involves Fock spaces constructed from R_{ii}

Wall-Crossing When $\alpha = \partial W$

$$\left(g_{I\bar{J}}(\phi; s), W(\phi; s) \right) \qquad 0 \leq s \leq 1$$

$\mu_{ij}(s)$ is only piecewise constant: CFIV, CV 1991, 1992

Happens when two central charges are parallel.

CVWCF: Tells how $\mu_{ij}(s)$ jump.

Categorified CVWCF: Describe how the ***homotopy equivalence class of*** R_{ij} jumps.

Remarks

1. We claim that the homotopy equivalence class is physically meaningful so this is a well-posed question.

2. Moreover, the homotopy equivalence class of R_{ij} is a nontrivial refinement of μ_{ij}

Remarks

It is often said that the only thing we can hope to compute exactly in interacting, non-integrable QFTs are Witten indices.

For general LG(X, W) are interacting & non-integrable

So what we are doing here has some tension with this standard folklore.

Wall-Crossing Formula With Twisted Masses: 1/ 3

``Vacuum Groupoid Algebra'' :

For each $\gamma_{ij} \in \Gamma_{ij}$ introduce a variable $x_{\gamma_{ij}}$

$$x_{\gamma_{ij}} \, x_{\gamma_{kl}} = \delta_{jk} \; x_{\gamma_{ij}+\gamma_{jk}}$$

Wall-Crossing Formula With Twisted Masses: 2/3

$\forall \quad i \neq j \quad \& \quad \gamma_{ij} \in \Gamma_{ij}$ $\qquad S_{\gamma_{ij}} := 1 + \mu_{\gamma_{ij}} \, x_{\gamma_{ij}}$

$\forall \quad \gamma \in \Gamma \qquad K_\gamma := \sum_i \prod_k (1 - x_\gamma)^{-\mu_{k\gamma_{ii}}} x_{u_i}$ $\qquad u_i \in \Gamma_{ii}$

Additive identity

For any half-plane $\mathbb{H} \subset \mathbb{C}$

$S(\mathbb{H}) =: \prod_{Z_{\gamma_{ij}}, Z_\gamma \in \mathbb{H}} S_{\gamma_{ij}} K_\gamma$: Phase-ordered product

Wall-Crossing Formula With Twisted Masses: 3/3

Wall-crossing statement:

$S(\mathbb{H})$ is invariant

provided no BPS rays enter/leave the half-plane $\mathbb{H}$

Very similar to the mathematics of the 2d-4d WCF.

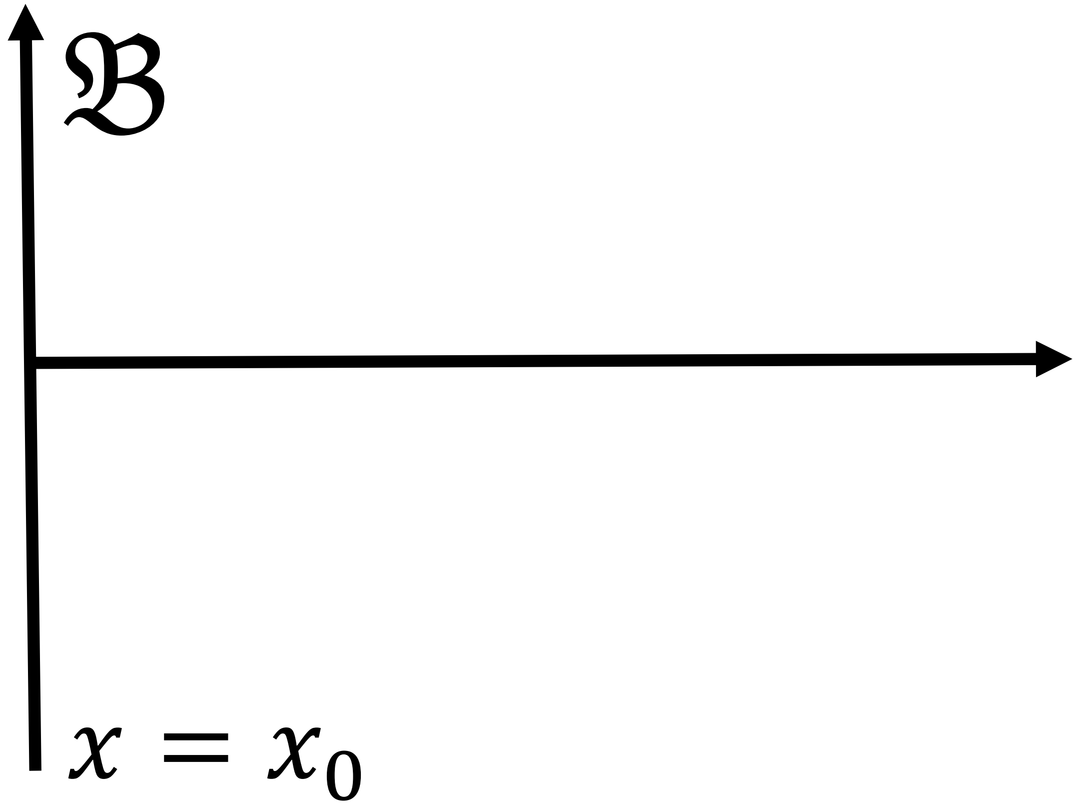
[Kontsevich,Soibelman 2008; Gaiotto,Moore,Neitzke 2010]

1. Supersymmetric Quantum Mechanics And Homological algebra
2. 2D N=(2,2) Landau-Ginzburg Models
3. Thimble Branes & Their Local Operators
4. Categorical Wall-Crossing
5. Generalization To Twisted Masses
6. Relation To 3d Indices

Branes

Use the (A_∞ algebra/category of) Branes.

The homotopy class of the category of branes is invariant.



Lefshetz Thimbles

Choose a half plane $\mathbb{H}$ and phase ζ .

For each vacuum ϕ_i there is a canonical brane $\mathfrak{T}_i$

Consider all values $\phi(x_0) \in X$ so there is a solution

$$\mathcal{L}_i^{right}(\zeta) \subset X \qquad \frac{\partial \phi^I}{\partial x} = i\,\zeta g^{I\bar{J}} \bar{\alpha}_{\bar{J}} \qquad \begin{array}{r} \phi(x_0) \to \phi_i \\ x \to +\infty \end{array}$$

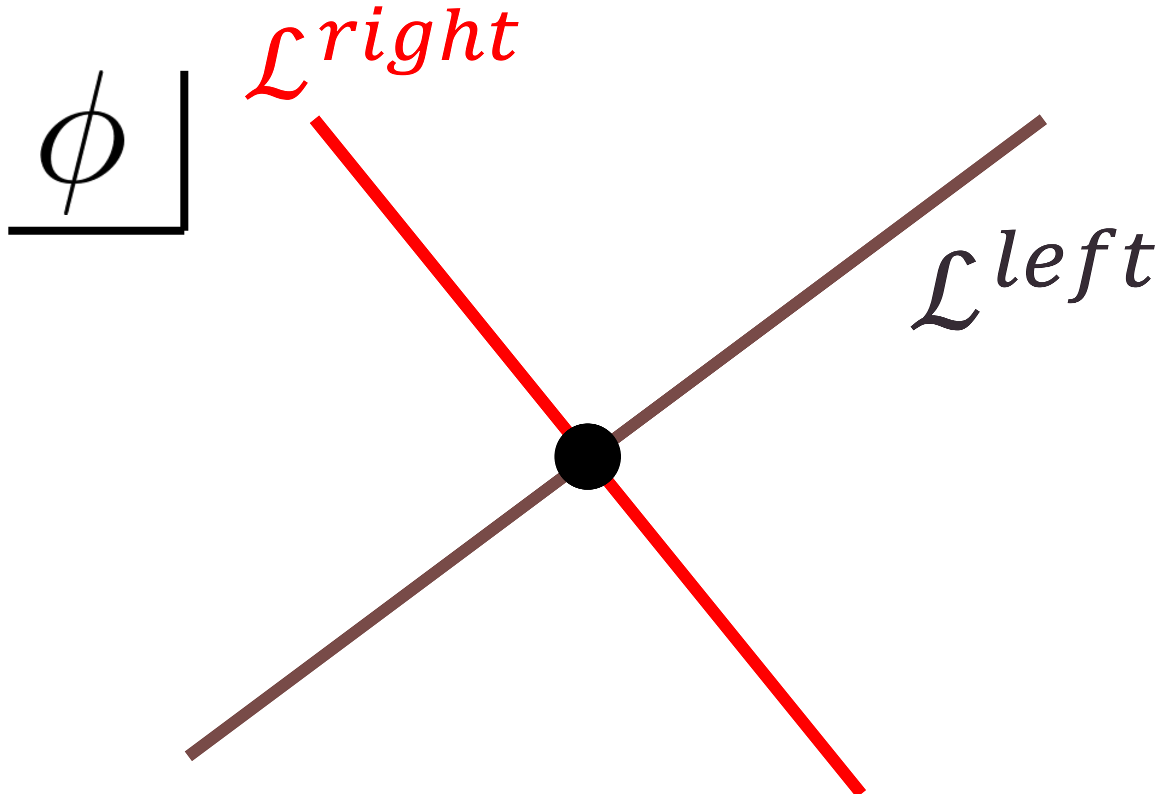
$\mathcal{L}_i^{right}(\zeta) \subset X$ are Lagrangian subspaces and provide nice half-susy bc's. [Hori-Iqbal-Vafa]

Example 1 of Lefshetz Thimbles

$$W = \frac{1}{2}\phi^2 \qquad \phi_i = 0 \qquad \frac{d\phi}{dx} = i\,\zeta\,\bar{\phi}$$

$$\phi(x) = c_0\sqrt{\pm i\zeta}\,e^{\pm x}$$

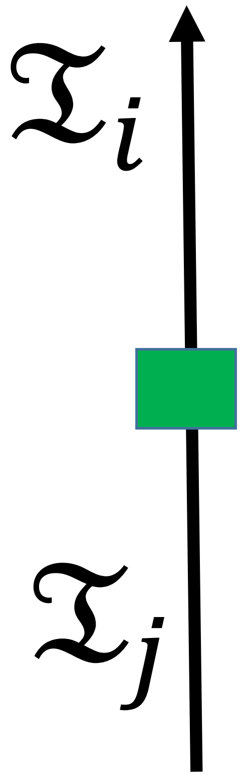
$$c_0 \in \mathbb{R}$$



Boundary Condition-Changing Local Operators

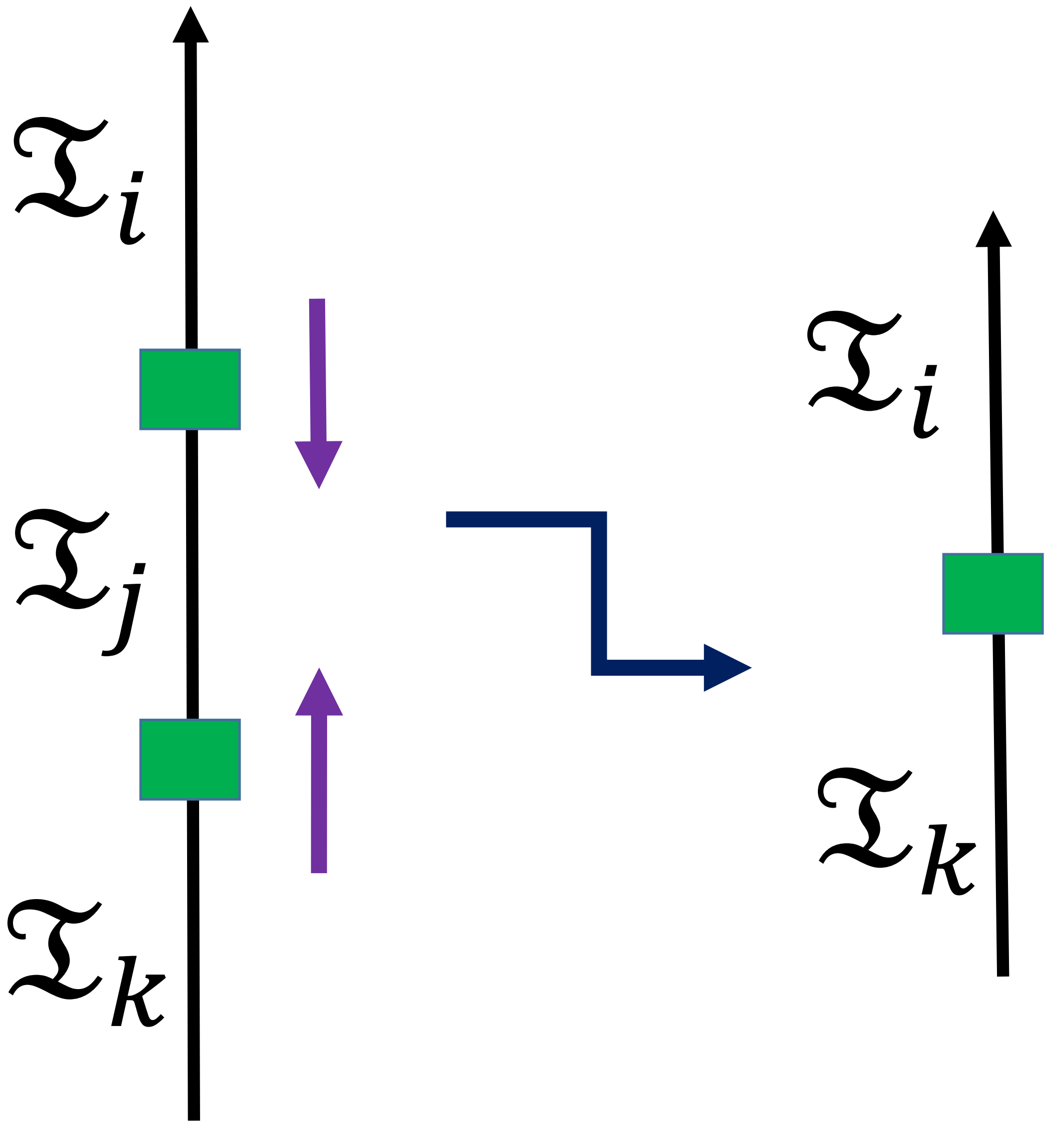
Choose a half-plane $\mathbb{H} \subset \mathbb{C}$, and a phase ζ

$\widehat{R}_{ij} :=$ Vector space of local bc changing operators between $\mathfrak{T}_j(\zeta)$ and $\mathfrak{T}_i(\zeta)$



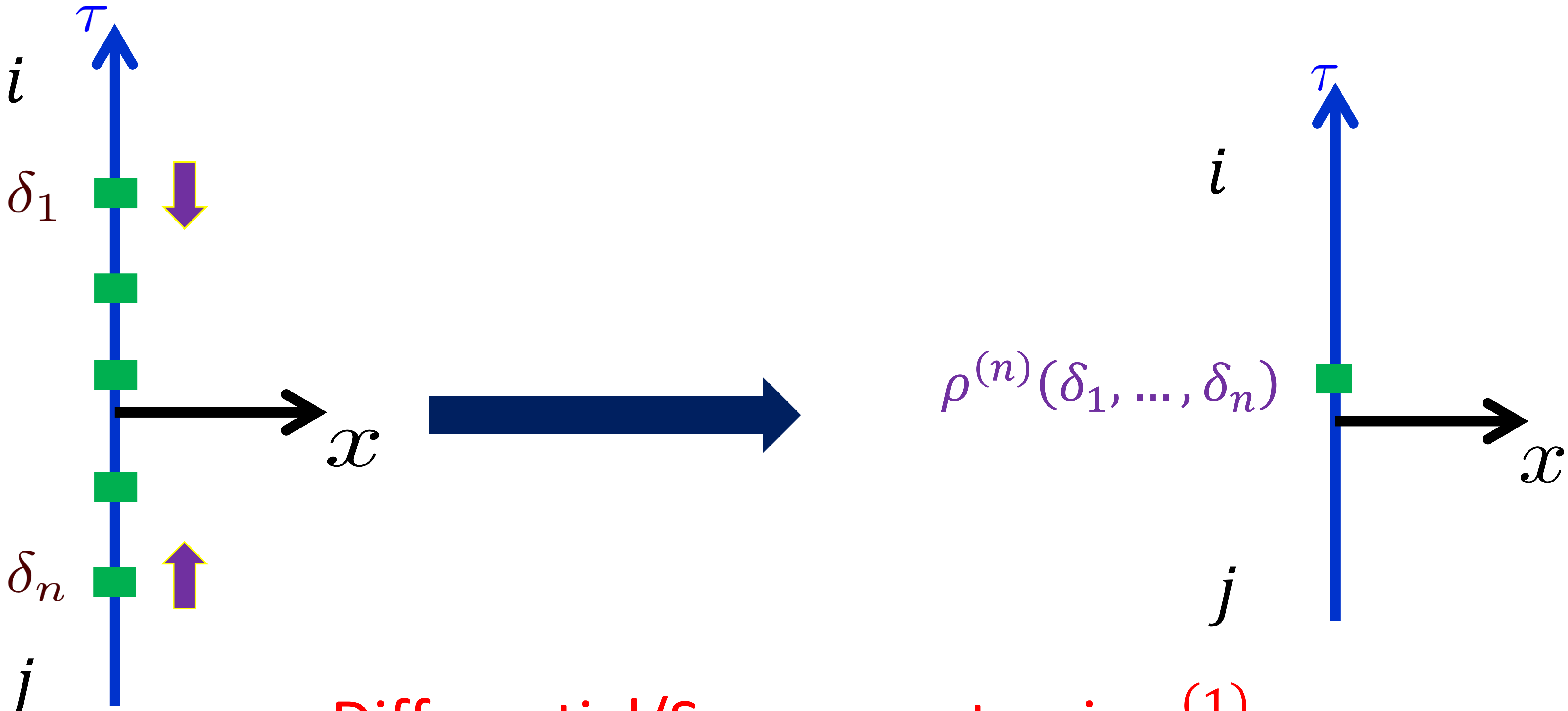
$\widehat{R}_{ij}$ is a chain complex

$\bigoplus_{ij} \hat{R}_{ij}$ Is An Algebra



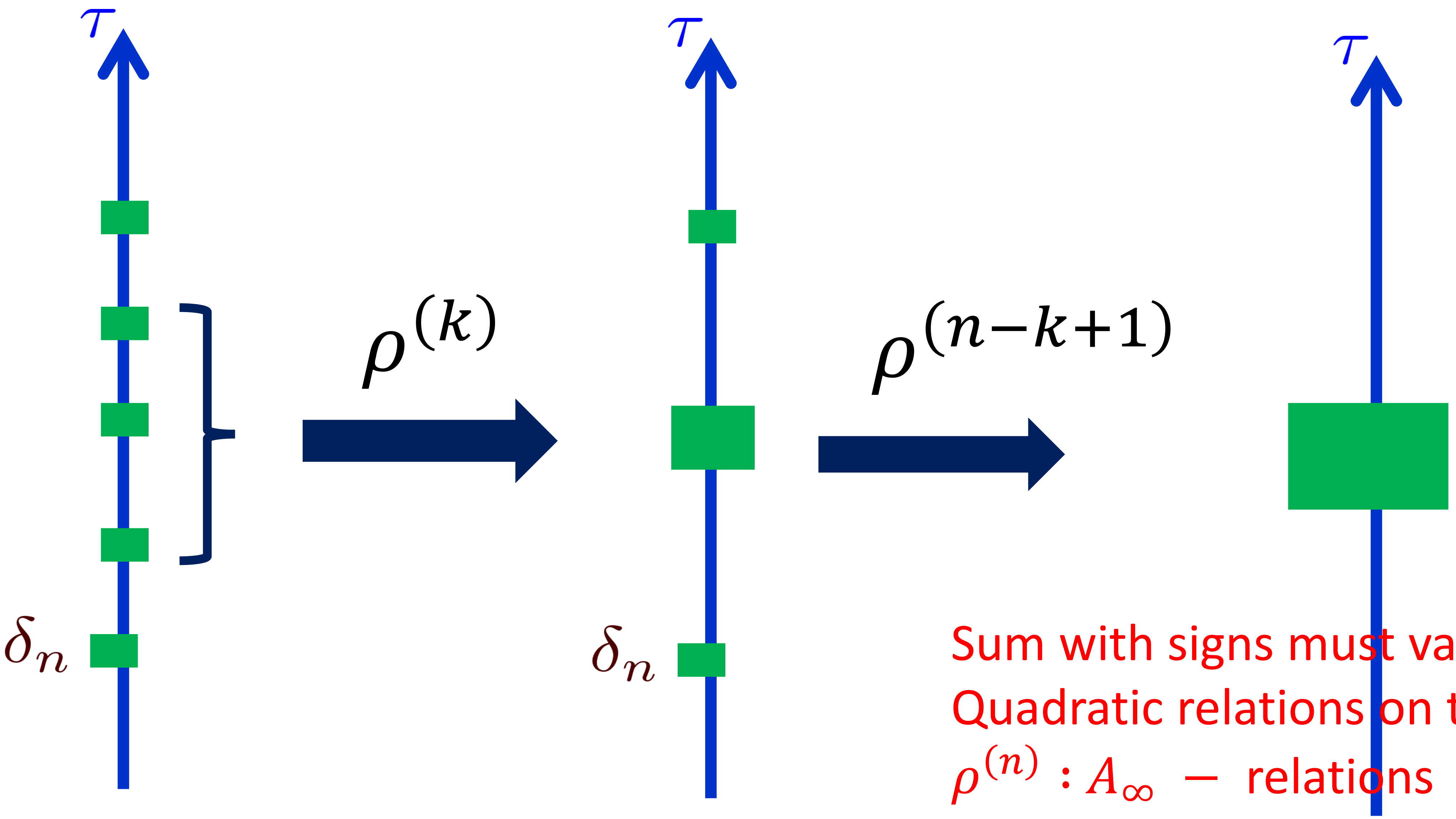
$$\rho^{(2)}_{ijk}: \hat{R}_{ij} \otimes \hat{R}_{jk} \to \hat{R}_{ik}$$

$\oplus_{ij}\ \hat{R}_{ij}$ has higher ``OPE Products''



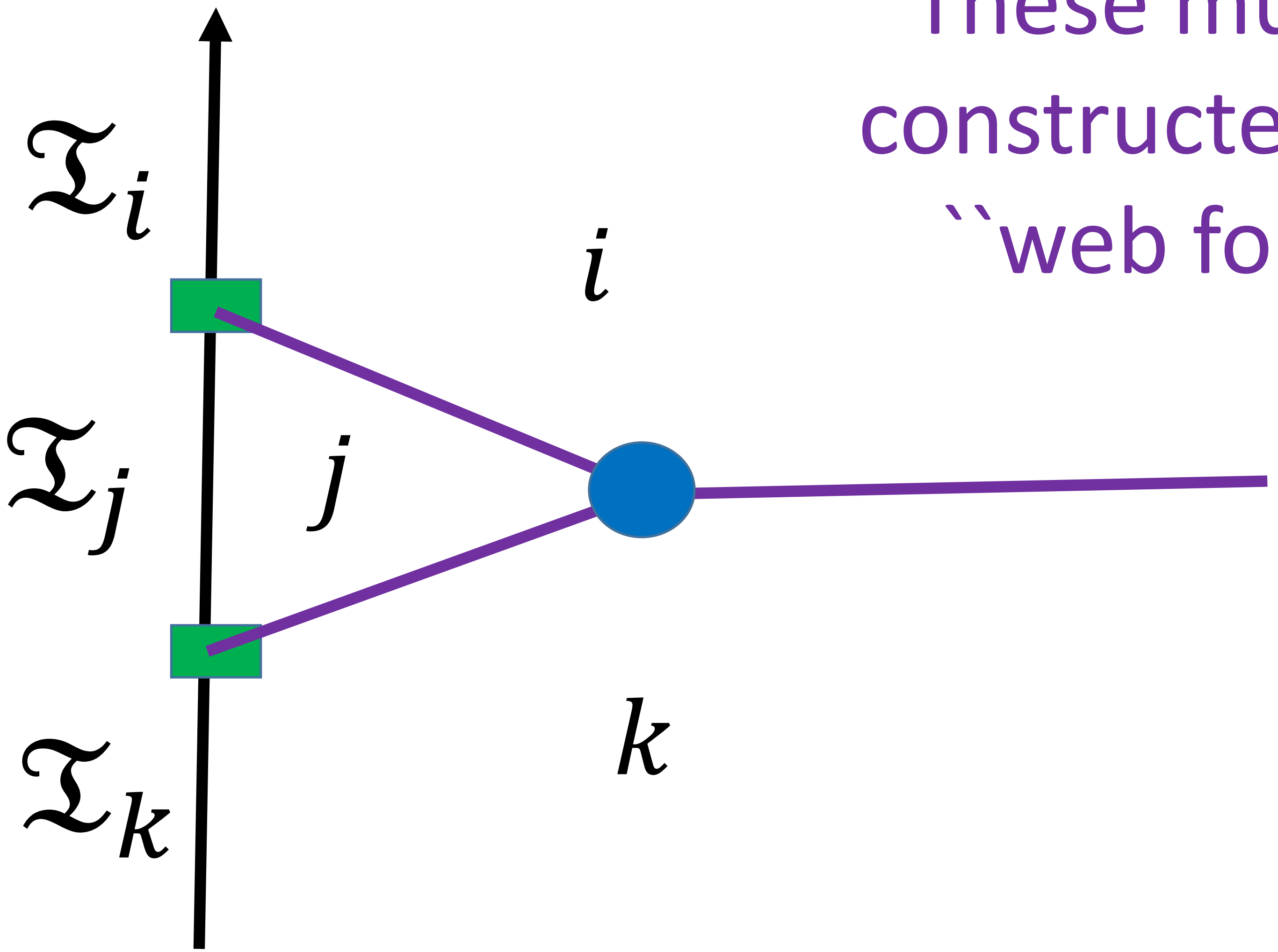
Differential/Susy operator is $\rho^{(1)}$

$\bigoplus_{ij} \hat{R}_{ij}$ Is An A_∞ −Algebra



Sum with signs must vanish:
Quadratic relations on the
$\rho^{(n)}$: A_∞ − relations

Web Formalism ($\alpha = \partial W$)



These multiplications can be constructed explicitly using the ``web formalism'' of GMW.

Sources For The Web Formalism

Algebra of the Infrared: String Field Theoretic Structures in Massive $\mathcal{N} = (2,2)$ Field Theory In Two Dimensions

---

Davide Gaiotto,[1] Gregory W. Moore,[2] and Edw itten[3]

Algebra of the infrared and secondary polytopes

M. apranov, M. Kontsevich, Y. Soibelman

August 13, 2014

Only 51 pages !!!

An Introduc sm

Davide Gaiotto,[1] Gregory W. Moore, Edward en[3]

There is a notion of homotopy equivalence of A_∞ -algebras.

It extends the notion of homotopy equivalence of chain complexes, and says how the OPE's are related to each other.

Categorical wall-crossing will involve the homotopy equivalence of these A_∞ —algebras.

An A_∞ − Category $\hat{R}$ (When $\alpha = \partial W$)

Choose $\zeta, \mathbb{H} \subset \mathbb{C}$

Objects = thimbles $\mathfrak{T}_i$

$$Hop(\mathfrak{T}_i, \mathfrak{T}_j) = \begin{cases} \hat{R}_{ij} & Z_{ij} \in \mathbb{H} \\ \mathbb{Z} & i = j \\ 0 & Z_{ij} \notin \mathbb{H} \end{cases}$$

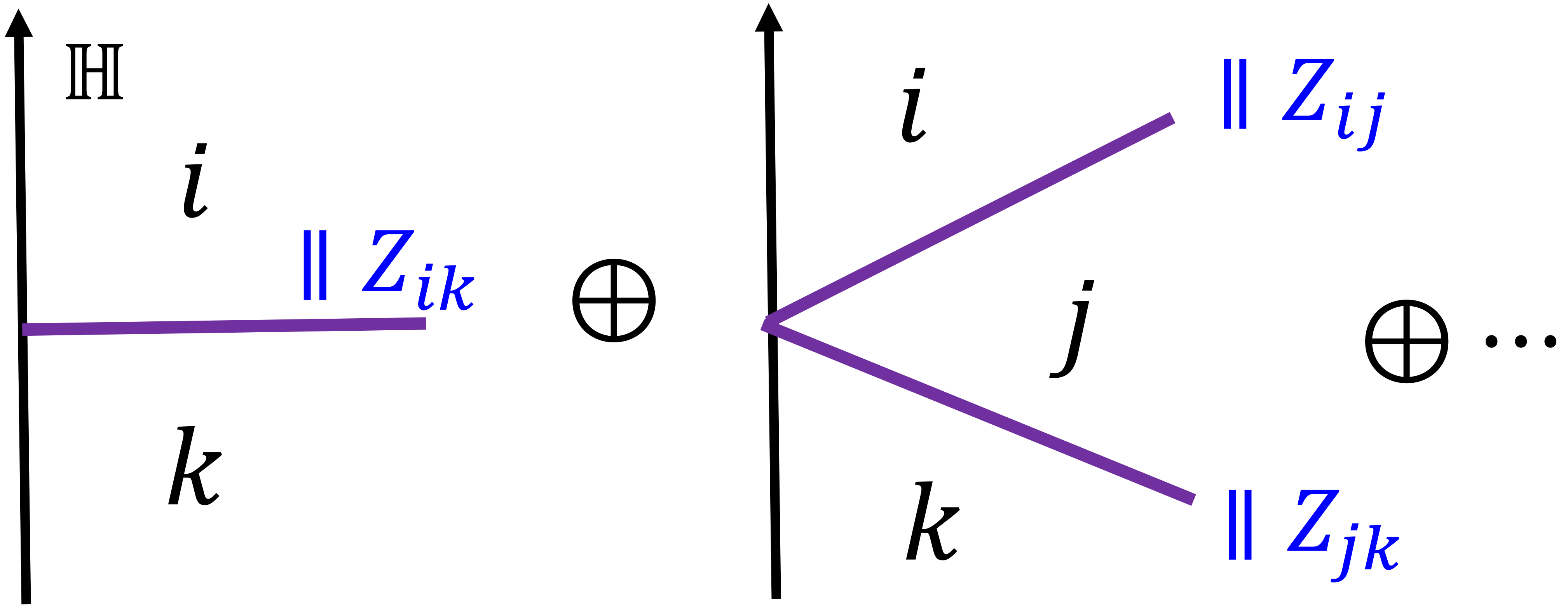
The Product Formula ($\alpha = \partial W$)

$\hat{R}_{ij}$ can be written in terms of R_{ij} [GMW]

$$\hat{R} := \oplus_{ij} \hat{R}_{ij} e_{ij} \qquad \underbrace{= \otimes_{z_{ij} \in \mathbb{H}}}_{\text{phase ordered!}} \left(\mathbb{Z}\, 1 + R_{ij} e_{ij} \right)$$

$$\hat{R}_{ik} = R_{ik} \oplus R_{ij} \otimes R_{jk} \oplus \cdots$$

$$\hat{R}_{ik} = R_{ik} \oplus R_{ij} \otimes R_{jk} \oplus \cdots$$



Summands correspond to sequences of central charges $Z_{i,i_1}, Z_{i_1,i_2}, \ldots, Z_{i_n,k}$ whose phases are ***clockwise ordered in the half plane $\mathbb{H}$***

Naïve Differential On $\hat{R}_{ik}$

$$\hat{R}_{ik} = R_{ik} \oplus R_{ij} \otimes R_{jk} \oplus \cdots$$

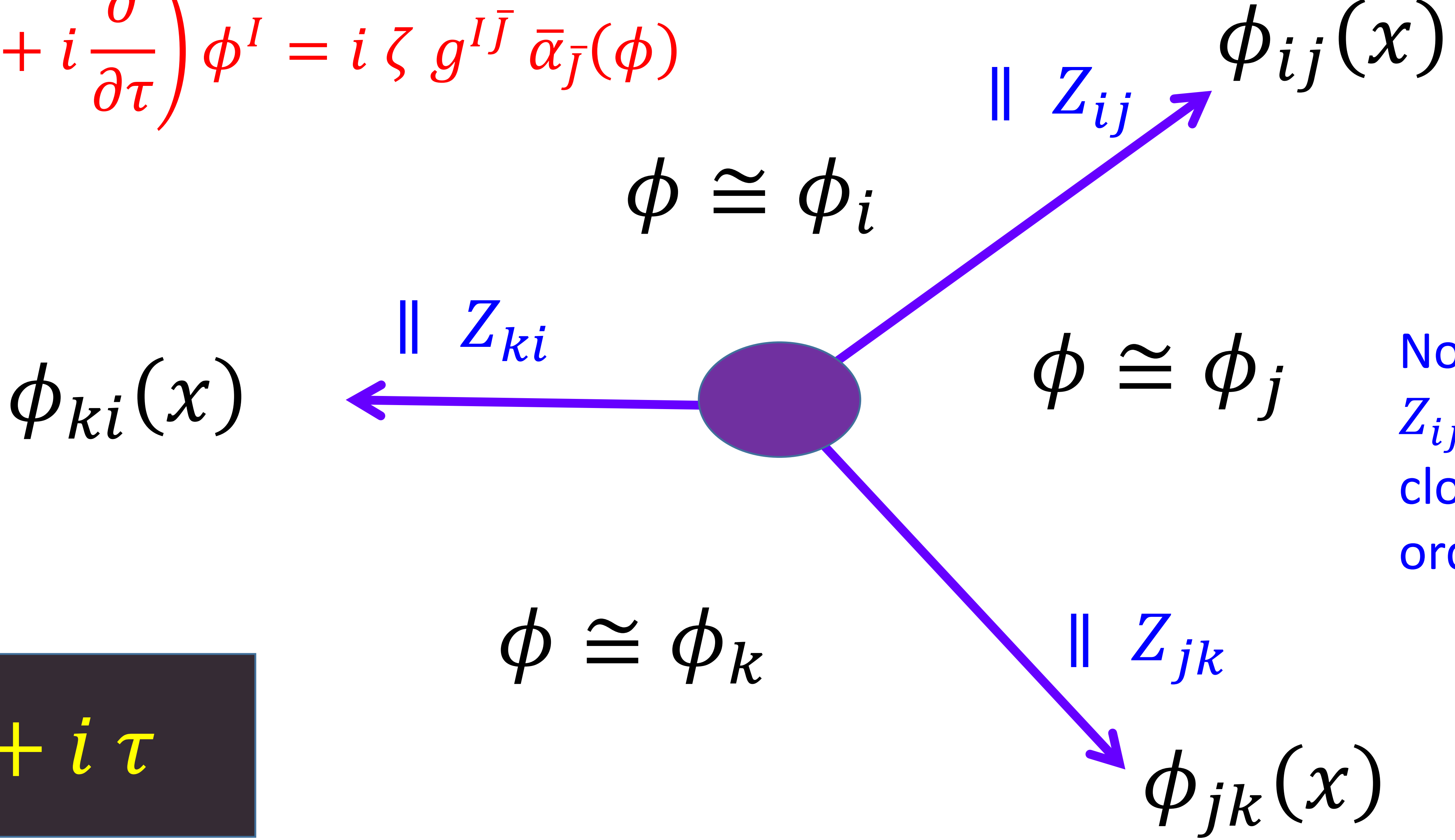
$$\hat{Q}^{naive} = Q_{ik} \oplus \left(Q_{ij} \otimes 1 + 1 \otimes Q_{jk} \right) \oplus \cdots$$

PHYSICALLY WRONG!

We missed important instanton effects

Domain Wall Junctions: ζ −instanton equation

$$\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial \tau}\right)\phi^I = i\,\zeta\, g^{I\bar{J}}\,\bar{\alpha}_{\bar{J}}(\phi)$$



Note:
Z_{ij}, Z_{jk}, Z_{ki} clockwise ordered….

$x + i\,\tau$

Explicit examples studied in

S. M. Carroll, S. Hellerman and M. Trodden, “Domain wall junctions are 1/4 - BPS states,” Phys. Rev. D 61, 065001 (2000) [hep-th/9905217].

G. W. Gibbons and P. K. Townsend, “A Bogomolny equation for intersecting domain walls,” Phys. Rev. Lett. 83, 1727 (1999) [hep-th/9905196].

H. Oda, K. Ito, M. Naganuma and N. Sakai, “An Exact solution of BPS domain wall junction,” Phys. Lett. B 471, 140 (1999) [hep-th/9910095].

Interior Amplitude

$$\phi_{ij} \otimes \phi_{jk} \otimes \phi_{ki} \in R_{ij} \otimes R_{jk} \otimes R_{ki}$$

Counting solutions defines an ``interior amplitude''

$$\beta_{ijk} \in R_{ij} \otimes R_{jk} \otimes R_{ki}$$

Summing over *all* such cyclic fans defines an L_∞ −algebra

$$R_c = \bigoplus_{cyclic\ fans} R_{i_1 i_2} \otimes \cdots \otimes R_{i_n i_1}$$

β is a Maurer-Cartan element in an L_∞ algebra.

(generalizes the ``broken flows identity.'')

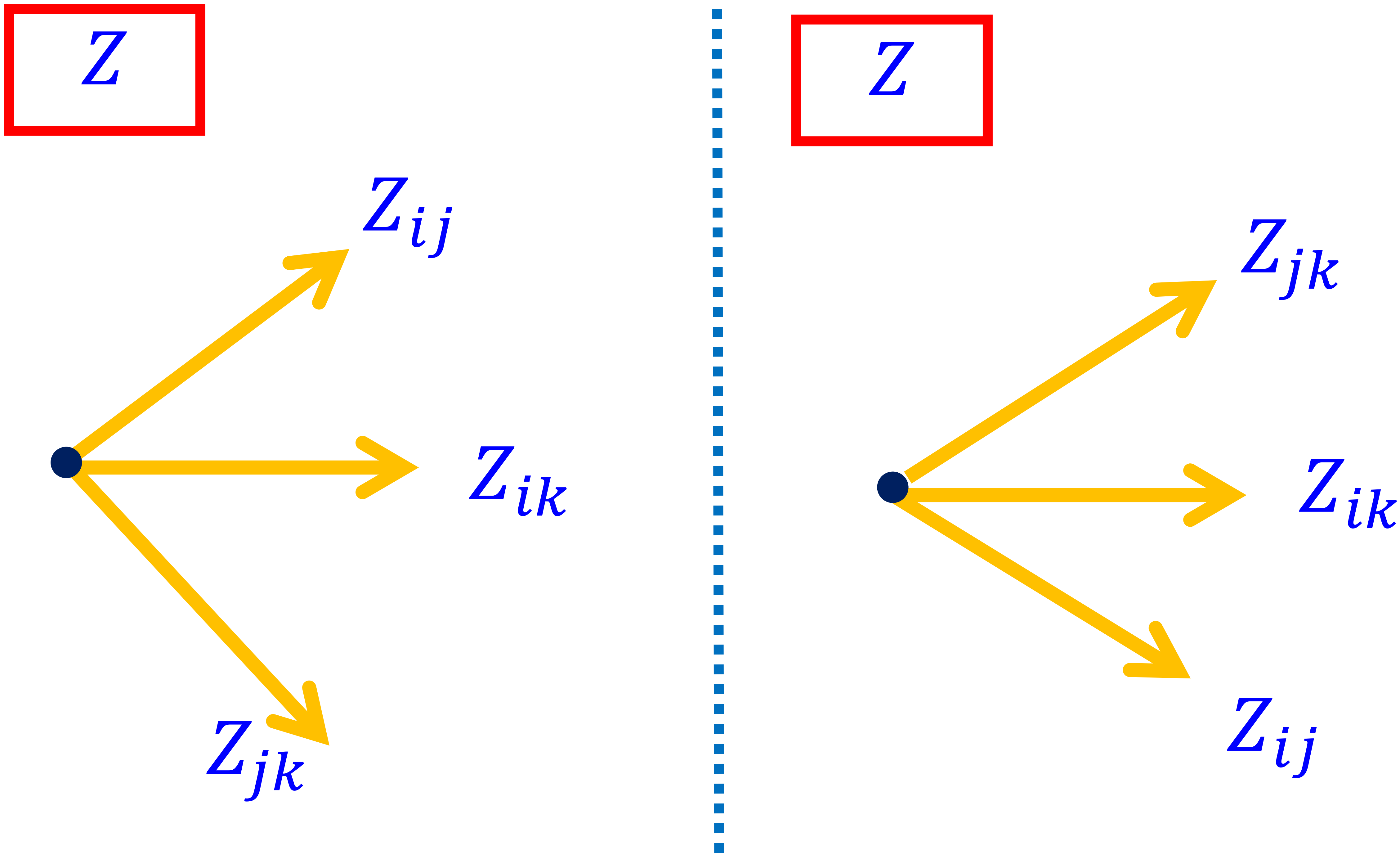
1. Supersymmetric Quantum Mechanics And Homological algebra
2. 2D N=(2,2) Landau-Ginzburg Models
3. Thimble Branes & Their Local Operators
4. Categorical Wall-Crossing
5. Generalization To Twisted Masses
6. Relation To 3d Indices

Categorical Wall-Crossing ($\alpha = \partial W$)

IF: $\left(X, g_{I\bar{J}}, \alpha, \zeta\right)_1 \sim \left(X, g_{I\bar{J}}, \alpha, \zeta\right)_2$

THEN: $\hat{R}\left(X, g_{I\bar{J}}, \alpha, \zeta\right)_1 \sim \hat{R}\left(X, g_{I\bar{J}}, \alpha, \zeta\right)_2$

$\Rightarrow$ how the R_{ij} complexes change (up to h.e.)



Z
Z_{ij}
Z_{ik}
Z_{jk}
Z
Z_{jk}
Z_{ik}
Z_{ij}

Definition: Cones In Homological Algebra

If $f : \mathcal{C}_1 \to \mathcal{C}_2$ is a chain map

$$\mathcal{C}_1 := (V_1, F_1, Q_1) \qquad \mathcal{C}_2 := (V_2, F_2, Q_2)$$

Then $Cone(f)$ is the new chain complex with

$$V := V_2 \oplus V_1[-1] \qquad Q_{Cone(f)} = \begin{pmatrix} Q_2 & f \\ 0 & -Q_1 \end{pmatrix}$$

Interior Amplitude Induces A Chain Map

$$\beta_{ijk} \in R_{ij} \otimes R_{jk} \otimes R_{ki}$$

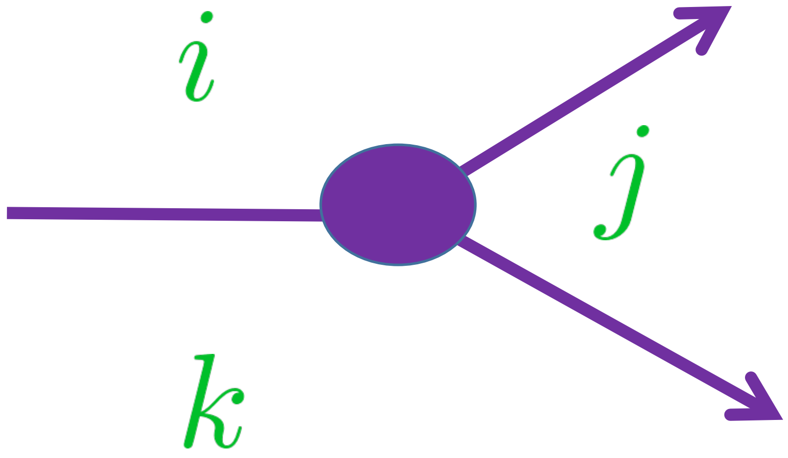
$CPT \Rightarrow \exists$ (deg=-1) contraction: $K: R_{ij} \otimes R_{ji} \to \mathbb{Z}$

Chain map:

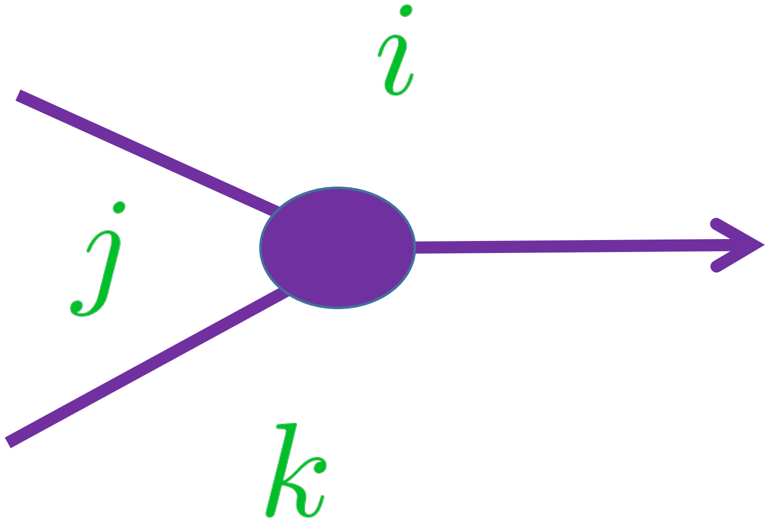
$$M(\beta_{ijk}): R_{ik} \to R_{ij} \otimes R_{jk}$$

Corrects $\hat{Q}^{naive}$: off-diagonal component of differential on $\hat{R}_{ik}$

$$M(\beta_{ijk}) = \hat{Q}_{ik}: R_{ik} \to R_{ij} \otimes R_{jk}$$



$Z_{ij} \parallel Z_{jk}$



Defines a chain map

$$M^L(\beta_{ijk}): R^L_{ik}[1] \to R^L_{ij} \otimes R^L_{jk}$$

Defines a chain map

$$M^R(\beta_{ikj}): R^R_{ij} \otimes R^R_{jk} \to R^R_{ik}$$

Solving Cat. Wall Crossing For $\alpha = \partial W$

An elegant way of solving the wall-crossing constraint

$$R_{ik}^{R} \sim Cone\left(M^{L}\left(\beta_{ijk}\right): R_{ik}^{L}[1] \rightarrow R_{ij}^{L} \otimes R_{jk}^{L}\right)$$

$$R_{ik}^{L} \sim Cone\left(M^{R}\left(\beta_{ikj}\right): R_{ij}^{R} \otimes R_{jk}^{R} \rightarrow R_{ik}^{R}\right)$$

Conversely, if $\hat{R}^{L} \sim \hat{R}^{R}$ then, up to homotopy,

R_{ik}^{L} and R_{ik}^{R}

are related by cone constructions as above

$$R_{ik}^{R} \sim Cone\left(M^{L}\left(\beta_{ijk}\right): R_{ik}^{L}[1] \rightarrow R_{ij}^{L} \otimes R_{jk}^{L}\right)$$

$$R_{ik}^{L} \sim Cone\left(M^{R}\left(\beta_{ikj}\right): R_{ij}^{R} \otimes R_{jk}^{R} \rightarrow R_{ik}^{R}\right)$$

Taking Euler characters gives the Cecotti-Vafa wall-crossing formula:

$$\mu_{ik}^{R} = \mu_{ik}^{L} + \mu_{ij}^{L}\,\mu_{jk}^{L}$$

$$\mu_{ik}^{L} = \mu_{ik}^{R} - \mu_{ij}^{R}\mu_{jk}^{R}$$

Cecotti-Vafa Cones

How do they generalize to the case with twisted masses?

1 Supersymmetric Quantum Mechanics And Homological algebra

2 2D N=(2,2) Landau-Ginzburg Models

3 Thimble Branes & Their Local Operators

4 Categorical Wall-Crossing

5 Generalization To Twisted Masses

6 Relation To 3d Indices

Generalization To Twisted Masses

Work in progress with Ahsan Khan

Definition: A ***cyclic fan of charges*** is a cyclically-ordered set

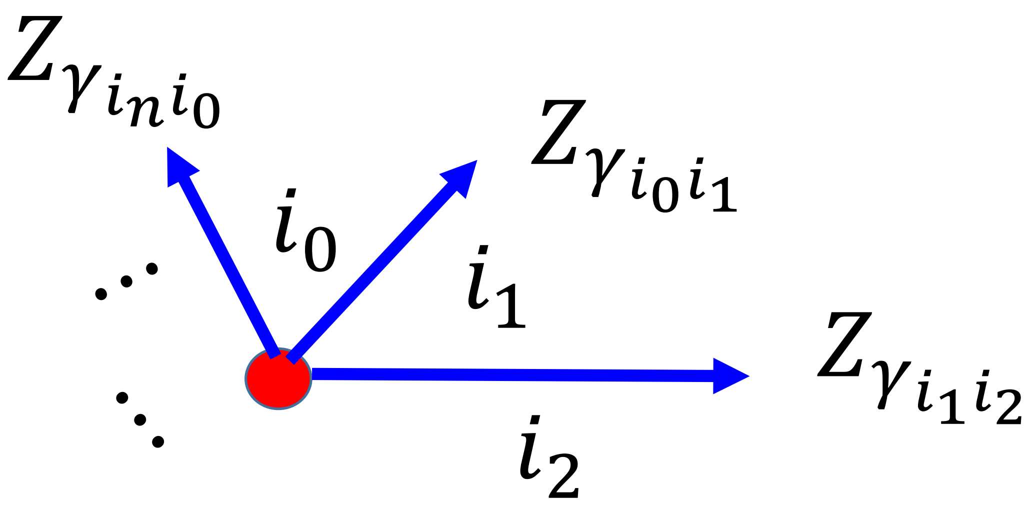
$$\{\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0}\}$$

So that the phases of $Z_{\gamma_{k,k+1}}$ are *monotonically* decreasing (clockwise)

New ingredient: Successive $Z_{\gamma_k, \gamma_{k+1}}$ can be parallel.

Representations Of (Irreducible) Webs

Irreducible fans: $\{\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0}\}$ $\quad i_k \neq i_{k+1}$

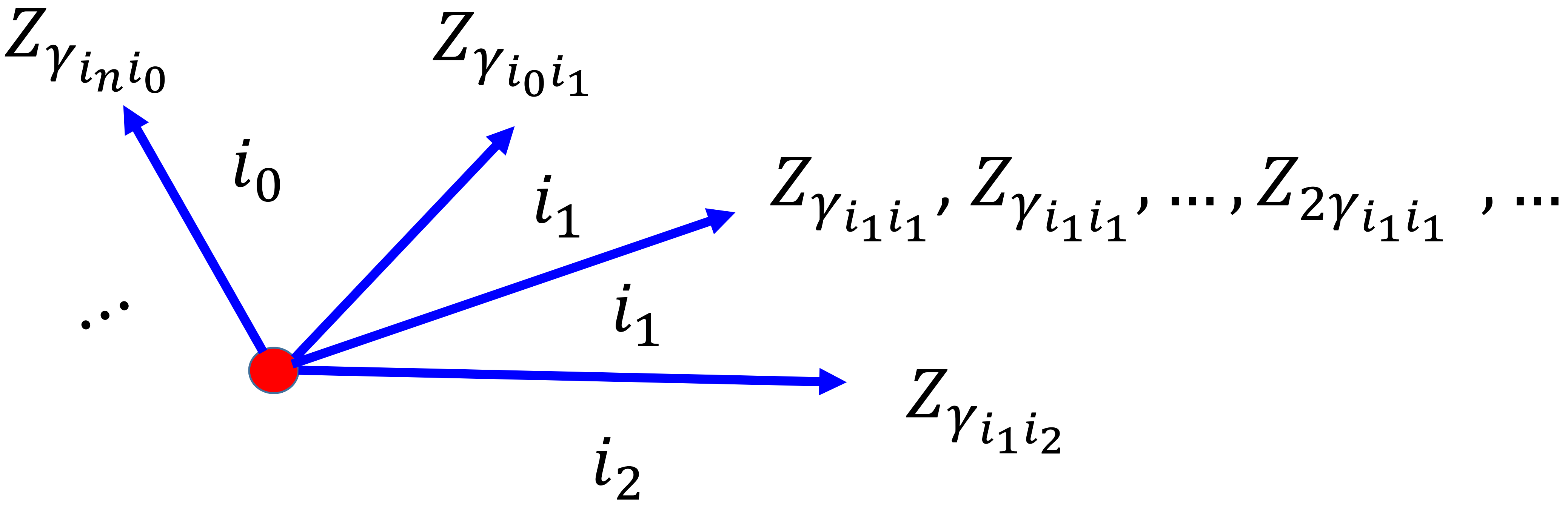


$$R_{\gamma_{i_0 i_1}} \otimes R_{\gamma_{i_1 i_2}} \otimes \cdots \otimes R_{\gamma_{i_n i_0}}$$

GMW web formalism applies to give L_∞ algebra structure on the sum over all these pictures

Vertices For Generalized Webs

$$\{\gamma_{i_0 i_1}, \gamma_{i_1 i_1}, \gamma_{i_1 i_1}, \dots, 2\gamma_{i_1 i_1}, \dots \quad \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0}\}$$

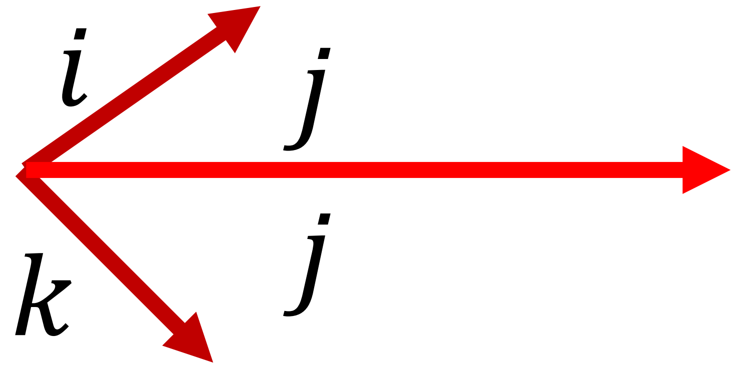


Representations Of Generalized Webs

$$\{\gamma_{i_0 i_1}, \gamma_{i_1 i_1}, \gamma_{i_1 i_1}, \dots, 2\gamma_{i_1 i_1}, \dots, \quad \gamma_{i_1 i_2}, \dots, \gamma_{i_n i_0}\}$$

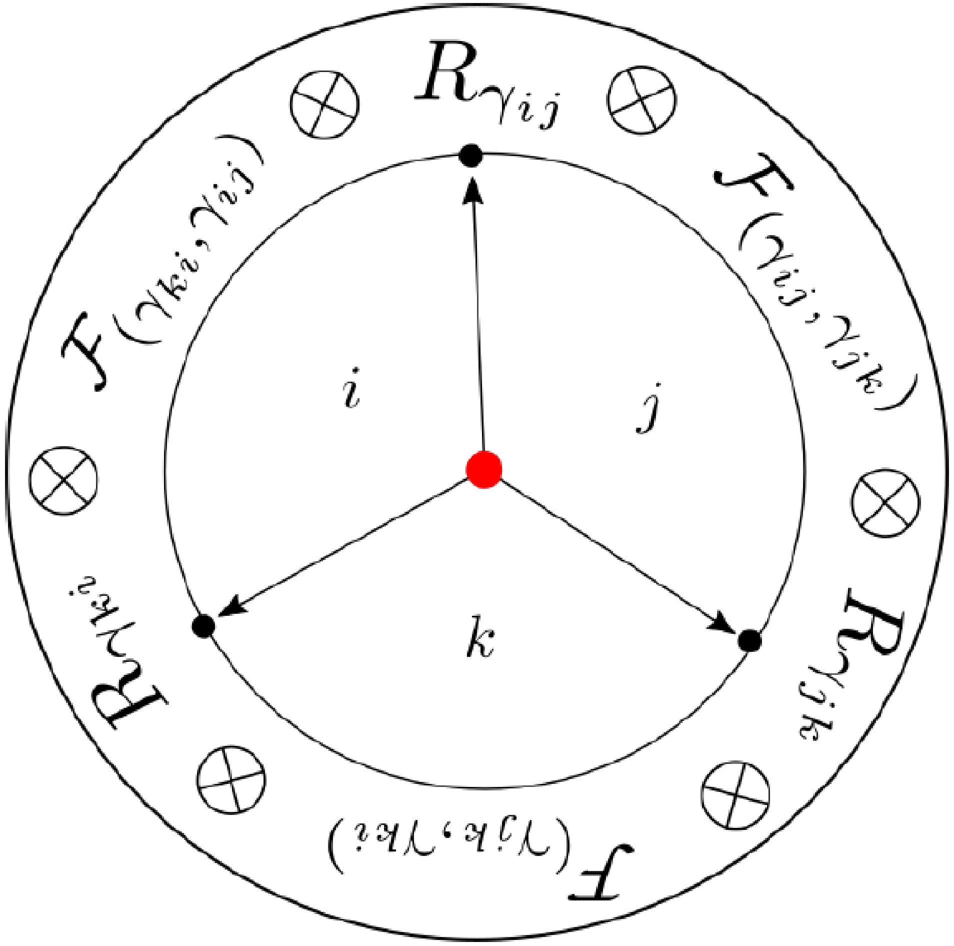
$$\mathcal{F}_{\gamma_{jj}} := \text{ Graded Fock space on } R_{\gamma_{jj}}$$

$$i \neq j \neq k \neq i \quad \mathcal{F}_{\gamma_{ij},\gamma_{jk}} := \bigotimes_{\gamma_{ij} < \gamma_{jj} < \gamma_{jk}} \mathcal{F}_{\gamma_{jj}}$$



$$R_c = \bigoplus_{\{\gamma_{i_0 i_1}, \dots, \gamma_{i_n i_0}\}\, irred}$$

$$\mathcal{F}_{\gamma_{i_n i_0}, \gamma_{i_0 i_1}} \otimes R_{\gamma_{i_0 i_1}} \otimes \cdots \otimes \mathcal{F}_{\gamma_{i_{n-1} i_n}, \gamma_{i_n i_0}} \otimes R_{\gamma_{i_n i_0}}$$



Conjecture: Has a natural L_∞ structure associated with generalized webs.

Checked in several special cases.

A_∞ − Category Of Branes

$\hat{R}_{ij}$:= Sum over all half-plane fans with
<u>Fock spaces of periodic jj solitons</u>
inserted between factors $R_{\gamma_{ij}}$ and $R_{\gamma_{jk}}$

Conjecture: $(\hat{R}, R_c)$ has the structure of an LA_∞-category (= open closed homotopy category)

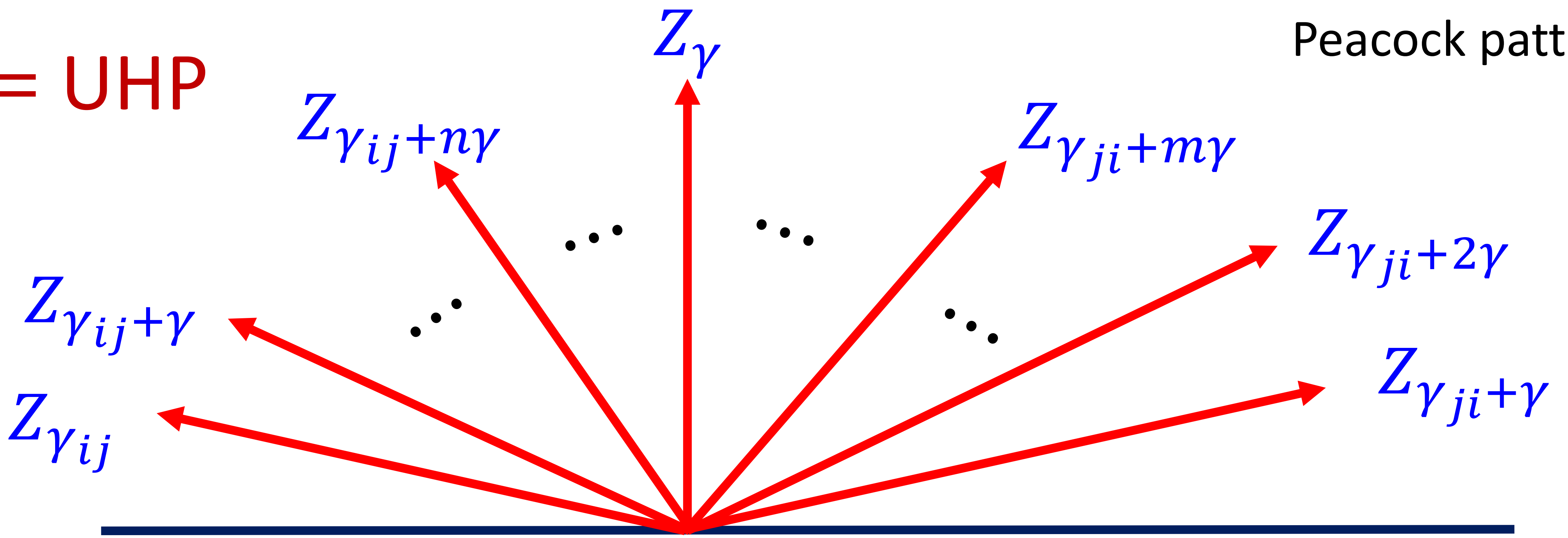
Example: Mirror Of $\mathbb{CP}^1$

$$\phi \in X = \mathbb{C}^* \qquad \alpha = \left(\frac{t}{\phi^2} + \frac{m}{\phi} + t\right) d\,\phi$$

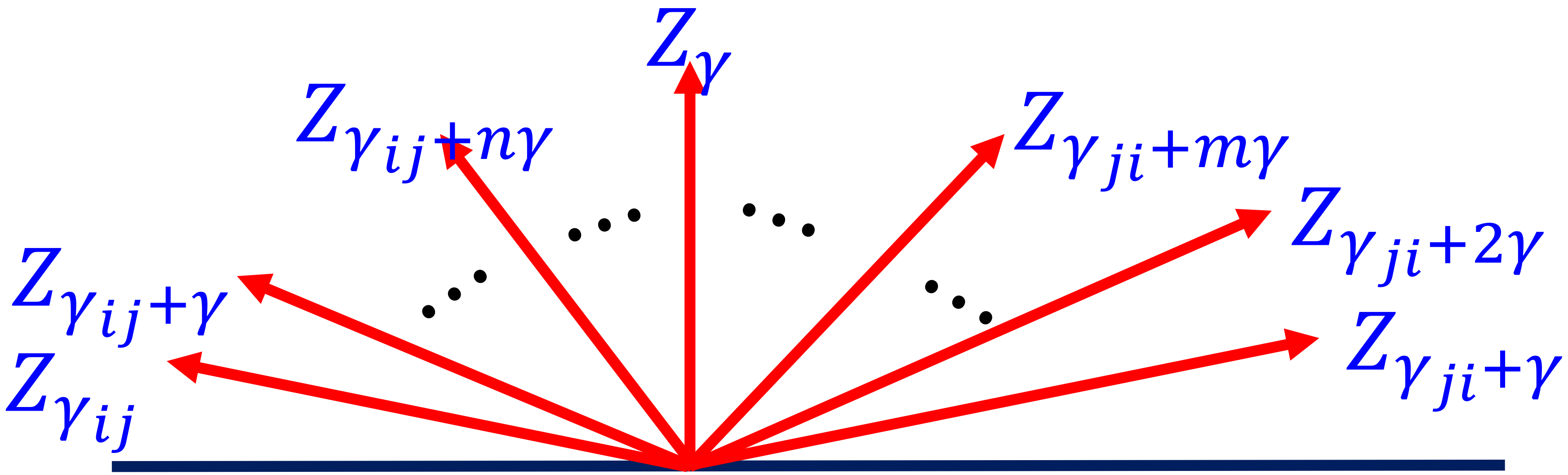
Two vacua: ϕ_i & ϕ_j & $\Gamma = \mathbb{Z}\gamma$ is rank one.

$\mathbb{H} = \mathrm{UHP}$

Peacock pattern



$\mathbb{H} = \mathrm{UHP}$



$$\hat{R} = \bigotimes_{(n=0)}^{\infty} \begin{pmatrix} \mathbb{Z} & R_{\gamma_{ij}+n\gamma} \\ 0 & \mathbb{Z} \end{pmatrix} \bigotimes_{k\geq 1} \begin{pmatrix} \mathcal{F}\left[R_{k\gamma_{ii}}\right] & 0 \\ 0 & \mathcal{F}\left[R_{k\gamma_{jj}}\right] \end{pmatrix} \bigotimes_{n=\infty}^{1} \begin{pmatrix} \mathbb{Z} & 0 \\ R_{\gamma_{ji}+n\gamma} & \mathbb{Z} \end{pmatrix}$$

1. Supersymmetric Quantum Mechanics And Homological algebra

2. 2D N=(2,2) Landau-Ginzburg Models

3. Thimble Branes & Their Local Operators

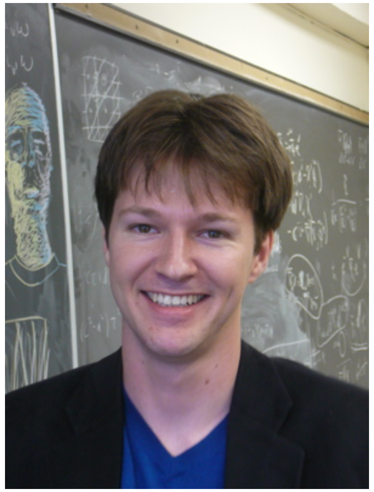
4. Categorical Wall-Crossing

5. Generalization To Twisted Masses

6. Relation To Three-Dimensional SQFT

RELATION TO 3D INDEX

With nontrivial input from Andy and Tudor



Motivation

Recent striking conjecture by Garoufalidis, Gu, and Marino,
``Peacock Patterns And Resurgence In Complex Chern-Simons Theory''

They observed a relation between the 3d index

$$I_T(q,x) = Tr_{\mathcal{H}_T}(-1)^F q^{\frac{1}{2}R+j_3}\, x^e$$

T: $3d$ SUSY class R theory associated
with a hyperbolic knot complement $T(M_K)$

and Stokes matrices related to
thimbles in complex Chern-Simons theory on M_K

We give it a natural context and state a conjecture about PDE’s (Kapustin-Witten equations) which implies the GGM conjecture.

In fact, the conjecture has been stated before by Victor Mikhaylov (2017) for different reasons.

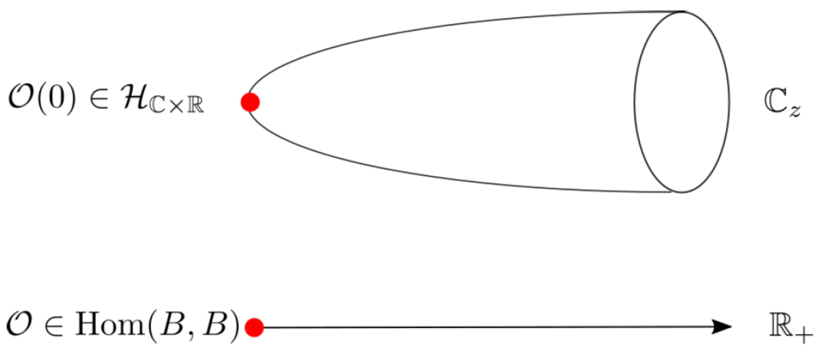
$T(M_3)$

$T(M_3)$ is a theory obtained by reduction of 6d (2,0) on M_3
with topological twist (class R)
[Dimofte, Gaiotto, Gukov; Terashima-Yamazaki]

Consider $T(M_3)$ on $\mathcal{C} \times \mathbb{R}_t$

With ``holomorphic-topological twist'' [Witten; Oh-Yagi]
we can identify the index $I_T(q, x)$ with the trace over the Q-cohomology of <u>*local*</u> operators at the tip of the cigar.

KK Reduction of Cigar ⇒ LG model With A Brane



Get a LG theory on half space with boundary condition $\mathfrak{B}_{Cigar}$

Q −closed local operators on $\mathcal{C} \times \mathbb{R}_t$

Boundary operators for $\mathfrak{B}_{cigar}$

$$I_{T(M_3)}(q,x) = Tr_{Hop(\mathfrak{B}_{cigar},\mathfrak{B}_{cigar})}(-1)^F q^{J_1} x^{J_2}$$

LG Model = CSLG$[M_3]$

$$
\begin{tikzcd}
 & 6d\ (2,0)/\ M_3 \times \mathcal{C} \times \mathbb{R}_t \arrow[dl, "M_3"'] \arrow[dr, "U(1)_{\mathcal{C}}"] & \\
T[M_3]/\ \mathcal{C} \times \mathbb{R}_t \arrow[d, "U(1)_{\mathcal{C}}"'] & & 5D\ SYM/\ M_3 \times \mathbb{R}_+ \times \mathbb{R}_t \arrow[d, "M_3"] \\
LG/\ \mathbb{R}_+ \times \mathbb{R}_t \arrow[rr, phantom, "\cong"] & & CSLG[M_3]/\ \mathbb{R}_+ \times \mathbb{R}_t \\
\mathfrak{B}_{cigar} & & \mathfrak{B}_{Nahm}
\end{tikzcd}
$$

Relation to KW Equations With Nahm bc's.

$6d\ (2,0)\ /\ M_3 \times \mathcal{C} \times \mathbb{R}_t \xrightarrow{U(1)_{\mathcal{C}}}$ 5d SYM / $M_3 \times \mathbb{R}_+ \times \mathbb{R}_t$

$$ds^2 = g_{IJ}(x) dx^I dx^J + dy^2 + dt^2$$

Witten: BPS equations = KW equations for $\mathcal{A}$ on $M_3 \times \mathbb{R}_+$

$$y \to 0 \qquad Im(\mathcal{A}^a) = \frac{e^a}{y} + \mathcal{O}(1) \qquad Re(\mathcal{A}^a) = \omega^a + \mathcal{O}(y)$$

e^a, ω^a : Dreibein and spin connection for Riemannian metric on M_3 & $G_c = SL(2, \mathbb{C})$

Nahm = $\sum$ Chan-Paton $\times$ Lefshetz

Vacua of $CSLG[M_3]$: Flat connections σ_i

$$\mathfrak{B}_{Nahm} \cong \sum_i \mathcal{E}_{\sigma_i} \mathfrak{T}_{\sigma_i}$$

KW equations with $\mathcal{A}_y = 0$ are the ζ –soliton equations for $CSLG[M_3]$

Chan-Paton complex $\mathcal{E}_{\sigma_i}(\mathfrak{B}_{Nahm})$ is MSW for KW & Nahm bc's @ $y \to 0$ & $\mathcal{A} \to \sigma_i$ @ $y \to \infty$

$$\mathcal{O} = (-1)^F q^{J_1} x^{J_2}$$

$$Tr_{Hop(\mathfrak{B}_{cigar},\mathfrak{B}_{cigar})}\mathcal{O}$$

$$= \sum_i Tr_{\mathcal{E}_{\sigma_i}}\mathcal{O} \times Tr_{Hop(\mathfrak{T}_{\sigma_i},\mathfrak{T}_{\sigma_j})}\mathcal{O} \times Tr_{\mathcal{E}_{\sigma_j}}\mathcal{O}$$

$$Tr_{Hop(\mathfrak{T}_{\sigma_i},\mathfrak{T}_{\sigma_j})}\mathcal{O} = S_{\sigma_i,\sigma_j}(q,x)$$

Stokes matrices for Chern-Simons thimbles

Specialize to Hyperbolic Knot Complement

$M_3 = M_K$ knot complement in S^3 of a hyperbolic knot.

Among flat $SL(2,\mathbb{C})$ connections on M_K there is a distinguished one: σ_1

σ_1: corresponds to the complete hyperbolic metric. $\mathcal{A} = \omega + i\, e$

Conjecture: $\mathcal{E}_{\sigma_i}(\mathfrak{B}_{Nahm}) \cong \begin{cases} \mathbb{Z} & \sigma_i = \sigma_1 \\ 0 & \text{Else} \end{cases}$

Interesting Generalization

L: Colored (by reps of SL(2)) link in M_3

Witten: Modify Nahm boundary condition in 5d SYM with 't Hooft line L

Corresponds to a brane $\mathfrak{B}(L)$ in $\mathrm{CSLG}[\mathrm{M}_3]$

Up to homotopy equivalence, it only depends on isotopy class of $L \subset M_3$

Potentially New Knot Invariants

Conjecture:

$a.)$ The h.e. class of the A_∞−category of $\mathfrak{Br}\big(CSLG(M_3)\big)$ is a 3-manifold invariant

b.) The h.e. class of A_∞ - algebras $Hop\big(\mathfrak{B}(L), \mathfrak{B}(L)\big)$ are (new?) colored link invariants.

Conclusion

Using the framework of GMW we derived a categorified version of the Cecotti-Vafa WCF

The framework can be extended to the case with twisted masses. Here there are qualitatively new features involving Fock spaces of periodic solitons.

This circle of ideas applied to CSLG$[M_3]$ gives a natural framework for interpreting a recent striking conjecture of GGM, and moreover suggests potentially new colored link invariants.