# Two Projects Using Lattices, Modular Forms, String Theory & K3 Surfaces

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Herstmonceaux Castle, June 23, 2016

# Part I

# Desperately Seeking Moonshine

a project with Jeff Harvey

still in progress…

# Part II

# Holography & Zamolochikov Volumes Of Moduli Spaces of Calabi-Yau Manifolds

Time permitting ……

G. Moore, ``Computation Of Some Zamolodchikov Volumes, With An Application,'' arXiv:1508.05612

# Motivation

Search for a conceptual explanation of Mathieu Moonshine phenomena.

Eguchi, Ooguri, Tachikawa 2010

Proposal: It is related to the ``algebra of BPS states.''

Something like: M24 is a distinguished group of automorphisms of the algebra of spacetime BPS states in some string compactification using K3.

# String-Math, 2014

Today's story begins in Edmonton, June 11, 2014.
Sheldon Katz was giving a talk on his work with
Albrecht Klemm and Rahul Pandharipande

He was describing how to count BPS states for type II strings on a K3 surface taking into account the $so(4) = su(2) + su(2)$ quantum numbers of a particle in six dimensions.

Slide # 86 said ....

- Define the *refined K3 BPS invariants* $\mathrm{R}^{h\geq 0}_{j_L,j_R}$ by

$$\sum_{h=0}^{\infty}\sum_{j_L}\sum_{j_R}\mathrm{R}^{h}_{j_L,j_R}\,[j_L]_u[j_R]_y q^h =$$

$$\prod_{n=1}^{\infty}\frac{1}{(1-u^{-1}y^{-1}q^n)(1-u^{-1}yq^n)(1-q^n)^{20}(1-uy^{-1}q^n)(1-uyq}$$

where $[j]_x = x^{-j}+\ldots+x^{j}$

- For $h\leq 2$ the nonvanishing invariants are

$$\mathrm{R}^0_{0,0}=1\,,$$

$$\mathrm{R}^1_{0,0}=20\,,\quad \mathrm{R}^1_{\frac{1}{2},\frac{1}{2}}=1\,,$$

$$\mathrm{R}^2_{0,0}=231\,,\quad \mathrm{R}^2_{\frac{1}{2},\frac{1}{2}}=21\,,\quad \mathrm{R}^2_{1,1}=1$$

# Heterotic/Type II Duality

Het/T4 = IIA/K3

DH states: Perturbative heterotic BPS states ⟺ D4-D2-D0 boundstates

Roughly: Cohomology groups of the moduli spaces of objects in $D^b(K3)$ with fixed K-theory invariant and stable wrt a stability condition determined by the complexified Kahler class.

Aspinwall-Morrison Theorem: Moduli space of K3 sigma models:

$$O_{\mathbb{Z}}(II^{20;4})\backslash O_{\mathbb{R}}(20;4)/\left(O_{\mathbb{R}}(20)\times O_{\mathbb{R}}(4)\right)$$

# Heterotic Toroidal Compactifications

$$\mathbb{M}^{1,1+d} \times T^{8-d}$$

$$II^{24-d,8-d} \hookrightarrow \Gamma^{24-d;8-d} \subset \mathbb{R}^{24-d;8-d}$$

$$P = (P_L; P_R) \in \Gamma^{24-d;8-d}$$

Narain moduli space of CFT's:

$$O_{\mathbb{Z}}(II^{24-d;8-d}) \backslash O_{\mathbb{R}}(24-d;8-d) / \left(O_{\mathbb{R}}(24-d) \times O_{\mathbb{R}}(8-d)\right)$$

# Crystal Symmetries Of Toroidal Compactifications

Construct some heterotic string compactifications with large interesting crystallographic group symmetries.

$$G \subset \mathrm{Aut}(\Gamma^{24-d;8-d})$$

$$G = G_L \times G_R$$

$$G_L \subset O_{\mathbb{R}}(24-d) \qquad G_R \subset O_{\mathbb{R}}(8-d)$$

Then G is a crystal symmetry of the CFT:

Example: Weyl group symmetries of enhanced YM gauge theories.

These are NOT the kinds of crystal symmetries we want

# Conway Subgroup Symmetries

Start with a distinguished d=0 compactification:

$$\mathbb{M}^{1,1} \times T^8$$

$$\Gamma^{24,8} = (\Lambda; 0) \oplus (0; \Gamma_8)$$

Crystal symmetry:

$$\mathrm{Co}_0 \times W(E_8)$$

Note that $\mathrm{Co}_0$ is not a Weyl group symmetry of any enhanced Yang-Mills gauge symmetry.

Now ``decompactify''

# A Lattice Lemmino

$\mathfrak{F}_L \subset \Lambda \quad \& \quad \mathfrak{F}_R \subset \Gamma_8$ isometric of rank d

Then ***<u>there exists an even unimodular lattice with embedding</u>***

$$\Gamma^{24-d;8-d} \hookrightarrow \mathbb{R}^{24-d;8-d}$$

such that, if

$$G_L := \mathrm{Fix}(\mathfrak{F}_L) \subset \mathrm{Aut}(\Lambda) = \mathrm{Co}_0$$

$$G_R := \mathrm{Fix}(\mathfrak{F}_R) \subset \mathrm{Aut}(\Gamma_8) = W(E_8)$$

$\Gamma^{24-d,8-d}$ has crystallographic symmetry

$$G_L \times G_R \subset O(24-d) \times O(8-d)$$

# Easy Proof

Uses standard ideas of lattice theory.

$$\mathcal{D}_+(\mathfrak{F}_L^\perp) \cong \mathcal{D}_-(\mathfrak{F}_L) \;\cong\; \mathcal{D}_-(\mathfrak{F}_R) \cong \mathcal{D}_+(\mathfrak{F}_R^\perp)$$

$$\Gamma \subset (\mathfrak{F}_L^\perp)^\vee \oplus (\mathfrak{F}_R^\perp)^\vee \subset \mathbb{R}^{24-d;8-d}$$

$$\Gamma = \{(x,y) | \bar{x} \cong \bar{y}\}$$

$$g : (x; y) \mapsto (g_L x; g_R y) \qquad \overline{g_L x} = \overline{x} \qquad \overline{g_R y} = \overline{y}$$

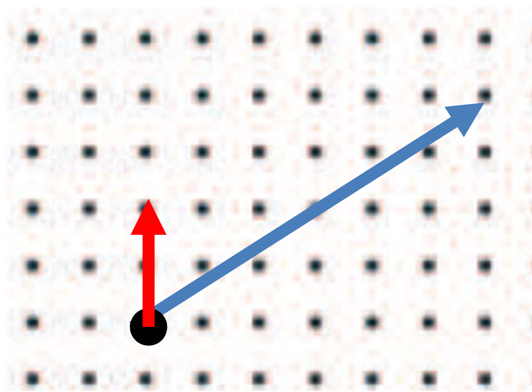
# CSS Compactifications

This construction defines points of moduli space with
**C**onway **S**ubgroup **S**ymmetry:
call these CSS compactifications.

What crystal symmetries can you get?

In general, a sublattice preserves none of the crystal symmetries of the ambient lattice.

Consider, e.g., the lattice generated by (p,q) in the square lattice in the plane.



# Fixed Sublattices Of The Leech Lattice

The culmination of a long line of work is the classification by Hohn and Mason of the 290 isomorphism classes of fixed-point sublattices of the Leech lattice:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 221 | 3 | 24 | $[2^3 3]$ (#3) | $8_3^{+3}$ | 0 | 4 | 16 | 1 | 1 | 3 | $\mathrm{Mon}_b$* |
| 222 | 2 | 9196830720 | $U_6(2)$ | $2_{II}^{-2}3^{+1}$ | 0 | 1 | 1 | 1 | 1 | - | S* |
| 223 | 2 | 898128000 | $McL$ | $3^{-1}5^{-1}$ | 1 | 1 | 1 | 1 | 1 | - | S* |
| 224 | 2 | 454164480 | $2^{10}.M_{22}$ | $4_2^{+2}$ | 0 | 1 | 1 | 1 | 1 | - | $\mathrm{Mon}_a$* |
| 225 | 2 | 44352000 | $HS$ | $2_2^{-2}5^{+1}$ | 0 | 1 | 1 | 1 | 1 | - | S* |
| 226 | 2 | 20643840 | $2^9.L_3(4).2$ | $4_1^{+1}8_1^{+1}$ | 0 | 1 | 2 | 1 | 1 | - | $\mathrm{Mon}_a$ |
| 227 | 2 | 10200960 | $M_{23}$ | $23^{+1}$ | 1 | 1 | 1 | 1 | 2 | 1 | $M_{23}$* |

# Symmetries Of D4-D2-D0 Boundstates

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 99 | 4 | 245760 | $[2^{12}].A_5$ | $2_{II}^{-2}4_{II}^{-2}$ | 0 | 1 | 1 | 1 | 1 | - | $\mathrm{Mon}_a$* |
| 100 | 4 | 30720 | $[2^9].A_5$ | $2_{II}^{-4}5^{-1}$ | 0 | 1 | 1 | 1 | 1 | - | $\mathrm{Mon}_a$* |
| 101 | 4 | 29160 | $3^4.A_6$ | $3^{+2}9^{+1}$ | 1 | 1 | 1 | 1 | 1 | - | S* |
| 102 | 4 | 20160 | $L_3(4)$ | $2_{II}^{-2}3^{-1}7^{-1}$ | 2 | 1 | 1 | 1 | 2 | 1 | $M_{23}$* |
| 103 | 4 | 12288 | $[2^{12}3]$ | $2_{II}^{+2}4_3^{+1}8_1^{+1}$ | 0 | 1 | 2 | 1 | 1 | - | $\mathrm{Mon}_a$ |
| 104 | 4 | 9216 | $[2^{10}3^2]$ | $2_{II}^{+4}3^{+2}$ | 0 | 1 | 2 | 1 | 1 | - | $\mathrm{Mon}_a$* |

These discrete groups will be automorphisms of the algebra of BPS states at the CSS points.

Het/II duality implies the space of D4D2D0 BPS states on K3 will naturally be in representations of these subgroups of $Co_0$ .

# Symmetries Of Derived Category

Theorem [Gaberdiel-Hohenegger-Volpato]: If $G \subset O_{\mathbb{Z}}(20;4)$ fixes a positive 4-plane in $\mathbb{R}^{20,4}$ then G is a subgroup of $Co_0$ fixing a sublattice with rank $\geq 4$.

Remark 1: GHV theorem classifies possible symmetries of sigma models with K3 target.

Remark 2: GHV generalize the arguments in Kondo's paper proving Mukai's theorem that the symplectic automorphisms of K3 are subgroups of M23 with at least 5 orbits on $\Omega$

Interpreted by Huybrechts in terms of the bounded derived category of K3 surfaces

$$G \cong \mathrm{Aut}_{H^{2,0}\oplus H^{0,2}}(D^b(K3)) \cap \mathrm{Aut}_{B+\mathrm{i}J}(D^b(K3))$$

# But Is There Moonshine In KKP Invariants?

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 99 | 4 | 245760 | $[2^{12}].A_5$ | $2_{II}^{-2}4_{II}^{-2}$ | 0 | 1 | 1 | 1 | 1 | - | $\mathrm{Mon}_a$* |
| 100 | 4 | 30720 | $[2^9].A_5$ | $2_{II}^{-4}5^{-1}$ | 0 | 1 | 1 | 1 | 1 | - | $\mathrm{Mon}_a$* |
| 101 | 4 | 29160 | $3^4.A_6$ | $3^{+2}9^{+1}$ | 1 | 1 | 1 | 1 | 1 | - | S* |
| 102 | 4 | 20160 | $L_3(4)$ | $2_{II}^{-2}3^{-1}7^{-1}$ | 2 | 1 | 1 | 1 | 2 | 1 | $M_{23}$* |
| 103 | 4 | 12288 | $[2^{12}3]$ | $2_{II}^{+2}4_3^{+1}8_1^{+1}$ | 0 | 1 | 2 | 1 | 1 | - | $\mathrm{Mon}_a$ |
| 104 | 4 | 9216 | $[2^{10}3^2]$ | $2_{II}^{+4}3^{+2}$ | 0 | 1 | 2 | 1 | 1 | - | $\mathrm{Mon}_a$* |

$$2^{12} : A_5 \cong 2^8 : M_{20}$$

So the invariants of KKP will show ``Moonshine'' with respect to this symmetry……

But this is a little silly: All these groups are subgroups of O(20). If we do not look at more structure (such as the detailed lattice of momenta/characteristic classes) we might as well consider the degeneracies as O(20) representations.

# Silly Moonshine

$$\prod_{n=1}^{\infty}\left(\frac{1}{(1-q^n)^{20}(1-yzq^n)(1-yz^{-1}q^n)(1-y^{-1}zq^n)(1-y^{-1}z^{-1}q^n)}\right)$$

is just the SO(4) character of a Fock space of 24 bosons.

$$\iota : O(20)\times O(4) \hookrightarrow O(24)$$

$$\iota^*(V) = \mathbf{20}\otimes\mathbf{1}\oplus\mathbf{1}\otimes\mathbf{4} \qquad \mathcal{F}_q(\mathbf{20}\otimes\mathbf{1}\oplus\mathbf{1}\otimes\mathbf{4})$$

$$\mathcal{F}_q(V) := \mathrm{Sym}_q^*(V)\otimes \mathrm{Sym}_{q^2}^*(V)\otimes\cdots$$

All the above crystal groups are subgroups of O(20) so the ``Moonshine'' wrt those groups is a triviality.

IS THERE MORE GOING ON ??

# Baby Case: T7 & d=1

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 272 | 2 | 80 | $[2^4 5]$ (#34) | $4_2^{+2}5^{+2}$ | 0 | 1 | 8 | 1 | 1 | - | $\mathrm{Mon}_a$ |
| 273 | 1 | $\lvert Co_2\rvert$ | $Co_2$ | $4_1^{+1}$ | 0 | 1 | 1 | 1 | 1 | - | S* |
| 274 | 1 | $\lvert Co_3\rvert$ | $Co_3$ | $2_3^{-1}3^{-1}$ | 0 | 1 | 1 | 1 | 1 | - | S* |

Decompose partition function of BPS states wrt reps of transverse rotation group O(1)

$$\mathcal{F}_q(V_{23}\otimes\mathbf{T}\oplus\mathbf{1}\otimes\mathbf{S}) = \mathbf{1}\otimes\mathbf{T} \;\oplus\; q\,[V_{23}\otimes\mathbf{T}\oplus\mathbf{1}\otimes\mathbf{S}]$$

$$\oplus\; q^2\,[\mathbf{300}\otimes\mathbf{T}\oplus\mathbf{24}\otimes\mathbf{S}] \;\oplus q^3\,[\mathbf{2876}\otimes\mathbf{T}\oplus\mathbf{324}\otimes\mathbf{S}]$$

$\oplus\cdots$ These numbers dutifully decompose nicely as representions of $Co_2$:

That's trivial because $Co_2 \subset O(23)$

$$\mathbf{300}\cong S^2V_{23}\oplus V_{23}\oplus\mathbf{1}$$

But is there a $Co_0$ x O(1) symmetry? $Co_0$ is *NOT* a subgroup of O(23). $Co_0$ x O(1) symmetry CANNOT come from a linear action on $V_{24}$.

# The SumDimension Game

$$\mathrm{Irrep}(Co_0) = \{\mathbf{1}, \mathbf{24}, \mathbf{276}, \mathbf{299}, \mathbf{1771}, \mathbf{2024}, \mathbf{2576}, \mathbf{4576}, \dots\}$$

$$\mathbf{1} \otimes \mathbf{T} \ \oplus \ q\,[V \otimes \mathbf{T} \oplus \mathbf{1} \otimes \mathbf{S}] \oplus q^2\,[\mathbf{300} \otimes \mathbf{T} \oplus \mathbf{24} \otimes \mathbf{S}] \ \oplus$$
$$q^3\,[\mathbf{2876} \otimes \mathbf{T} \oplus \mathbf{324} \otimes \mathbf{S}] \ \oplus \cdots$$

$$\mathbf{300} = \mathbf{299} + \mathbf{1} \qquad \mathbf{300} = \mathbf{276} + \mathbf{24}$$

$$\mathbf{2876} = \mathbf{2576} + \mathbf{299} + \mathbf{1} \qquad \mathbf{2876} = \mathbf{2576} + \mathbf{276} + \mathbf{24}$$

$$\mathbf{324} = \mathbf{299} + \mathbf{24} + \mathbf{1} \qquad \mathbf{324} = \mathbf{276} + \mathbf{24} + \mathbf{24}$$

ETC.

# Defining Moonshine

Any such decomposition defines the massive states of $\mathcal{F}_q(V)$ as a representation of $Co_0 \times O(1)$.

Problem: There are infinitely many such decompositions! What physical principle distinguishes which, if any, are meaningful?

Definition: You have committed ***Moonshine*** (for d=1) if you exhibit the massive sector of $\mathcal{F}_q(V)$ as a representation of $Co_0 \times O(1)$such that the graded character of any element g:

$$\mathrm{Tr}_{\mathcal{F}_q(V)} g q^{L_0 - 1}$$

is a modular form for $\Gamma_0(m)$ where m = order of g .

# Virtual Representations

Most candidate $Co_0$ x O(1) representations will fail to be modular.

But if we allow *virtual* representations:

$$V_{23} \to V_{24} - \mathbf{1}$$

$$\mathcal{F}_q(V_{23} \otimes \mathbf{T} \oplus \mathbf{1} \otimes \mathbf{S}) \to \mathcal{F}_q(V_{24}) \otimes \frac{\mathcal{F}_q(\mathbf{1} \otimes \mathbf{S})}{\mathcal{F}_q(\mathbf{1} \otimes \mathbf{T})}$$

The characters are guaranteed to be modular!

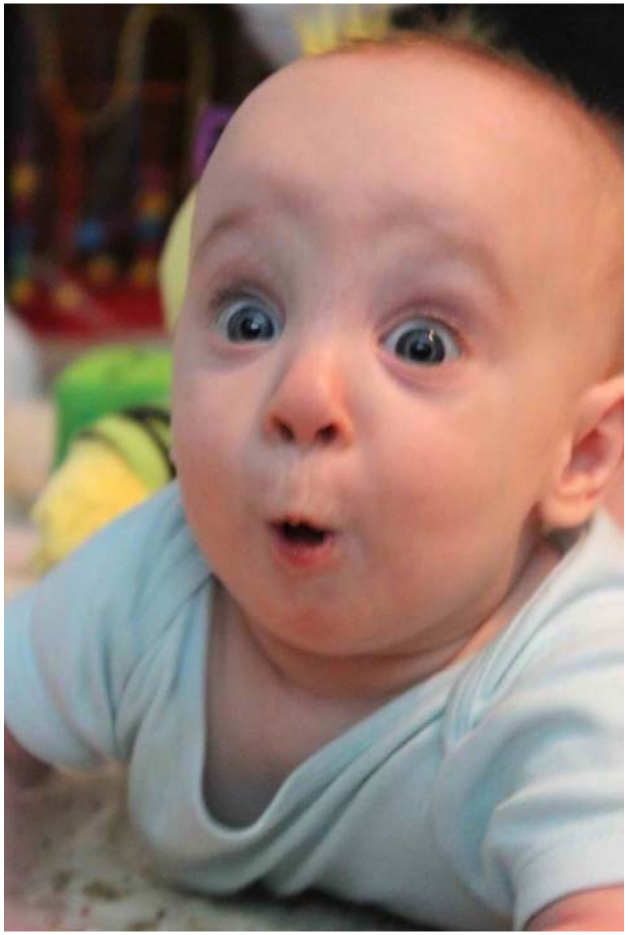
But massive levels will in general be *virtual* representations, not true representations.

# But, There Can Be Magic …

If the negative representations cancel for ALL the massive levels then there is in fact a modular invariant solution to the SumDimension game.

In fact, the negative representations of $Co_0 \times O(1)$ do indeed cancel and ALL the massive levels are in fact true representations!!

Even though there is no linear representation of $Co_0 \times O(1)$ on the 24 bosons that gives the above degeneracies….



But! The same argument also shows they are also true representations of O(24) x O(1).

# Lessons

Modularity of characters is crucial.

Virtual Fock spaces are modular.

There <u>can</u> be nontrivial cancellation of the negative representations.

A ``mysterious'' discrete symmetry can sometimes simply be a subgroup of a more easily understood continuous symmetry.

# What About d=4 ?

$$\mathrm{Co}_0 \times O(1) \to M_{24} \times O(4)$$

$$\mathcal{F}_q(V_{20} \otimes \mathbf{1} \oplus \mathbf{1} \otimes V_4) \to \mathcal{F}_q(V_{23}) \otimes \frac{\mathcal{F}_q(\mathbf{1} \otimes V_4)}{\mathcal{F}_q(\mathbf{1} \otimes \mathbf{1})^3}$$

Magical positivity fails:

$$\dim \mathrm{R}^2_{0,0} = 231$$

$$\mathrm{R}^2_{0,0} = V_{252} - V_{23} + 2V_1$$

*But we are desperately seeking Moonshine....*

So we ask: Could it still be that, magically, some positive combination of representations from the SumDimension game is nevertheless modular?

So we played the sum dimension game in all possible ways for lowest levels – the possibilities rapidly proliferate….

For each such decomposition we calculated the graded character of involutions 2A and 2B in M24

The resulting polynomial in q is supposed to be the leading term of SOME modular form of SOME weight with SOME multiplier system….

# Characters Of An Involution

$$
\begin{aligned}
Z_{2A} &= 8 + 1/q + 36q + 144q^2 + 282q^3 \qquad + \ldots\ldots \\
&= 8 + 1/q + 36q + 144q^2 + 426q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 218q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 362q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 266q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 410q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 202q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 346q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 378q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 522q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 314q^3 \\
&= 8 + 1/q + 36q + 144q^2 + 458q^3
\end{aligned}
$$

Should be modular form for $\Gamma_0(2)$ .

Weight?

(assumed half-integral)

Multiplier system?

# A Trick

$\Gamma_0(2)$ is generated by T and $ST^2S$

$ST^2S$ has an ``effective fixed point''

$$ST^2S \cdot \tau = \frac{\tau}{1-2\tau}$$

$$\tau_0 = \frac{1}{2}(1+\mathrm{i}) \qquad (ST^2S)\tau_0 = \tau_0 - 1$$

One can deduce the multiplier system from the weight.

# What Is Your Weight?

$$\tau = \tau_0 + \delta\tau$$

$$ST^2S \cdot \tau = \tau_0 - 1 - \delta\tau$$

$$w = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \log \left| \frac{Z(\tau_0 - \mathrm{i}\epsilon)}{Z(\tau_0 + \mathrm{i}\epsilon)} \right|$$

$$q_0 = e^{2\pi \mathrm{i} \tau_0} = -e^{-\pi} = -0.04...$$

Convergence is good so can compute the weight numerically.
For $Z_{2A}$ it converges to -8.4...... Not pretty. Not half-integral !!

No positive combination of reps is modular. No M24 Moonshine.

# Application To Heterotic-Type II Duality

Existence of CSS points have some interesting math predictions.

$\mathfrak{X}$ K3 and ellipticaly fibered CY3

$$\mathrm{Het}/T^2 \times K3' \quad \Longleftrightarrow \quad \mathrm{IIA}/\mathfrak{X}$$

Perturbative heterotic string states $\Longleftrightarrow$ Vertical D4-D2-D0 boundstates

# Generalized Huybrechts Theorem

So, if we can make a suitable orbifold of CSS compactifications of the heterotic string on T6

$$T^6/G \cong T^2 \times K3$$

And if there is a type II dual then we can conclude:

$$\mathrm{Aut}_\sigma(D^b(\mathfrak{X})_{\mathrm{vertical}})$$

is the subgroup of $Co_0$ fixing the rank two sublattice and centralizing the orbifold action.

# An Explicit Example -1/2

For simplicity: $\mathbb{Z}_2$ orbifold

$$X \to RX + \delta$$

$$R = (g_L; g_R) \in G_L \times G_R$$

Gram matrix of $\mathfrak{F}_L \cong \mathfrak{F}_R = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

There exists $g_R$ in W(E8) fixing $\mathfrak{F}_R$ with ev's $+1^4$, $-1^4$. Mod out by this on the right.

# An Explicit Example – 2/2

Need to choose involution $g_L$.

$$g_L \sim \mathrm{Diag}\{-1^{12}, +1^{12}\}$$

Flips sign of 12 coordinates $x^i$ in a dodecad of the Golay code

Passes known, nontrivial, consistency checks. (Especially, a poorly understood criterion of Narain, Sarmadi & Vafa.)

$$G_L = 2^{10}.M_{10}$$

$$(h^{1,1}(\mathfrak{X}), h^{2,1}(\mathfrak{X})) = (11, 11)$$

# Three General Questions

## Question One

Should every heterotic model on K3 x $T^2$ have a type II dual?

## Question Two

In the type II interpretation CSS only arise for special values of the flat RR fields. For example – heterotic on $T^8$ can give $E_8{}^3$ gauge symmetry. Somehow IIA/K3 x $T^4$ must have such gauge symmetry! An extension of the derived category viewpoint should account for this.

# Question Three

There is a well-developed theory of D-brane categories on $\mathfrak{X}$: Derived category and Fukaya category, related by homological mirror symmetry.

What about D-brane categories on $\mathfrak{X} \times S^1$ ?

D-branes sit at a point of $S^1$ or wrap it.

$\Rightarrow$ $D(\mathfrak{X}) + Fuk(\mathfrak{X})$.

But there can be boundstates between them!

At self-dual radius of $S^1$ there is N=3 susy!

So it should be possible to study this using topological sigma models.

# Part I Conclusions

So, what can we say about Mathieu Moonshine?

GHV: Quantum Mukai theorem:
It is not symmetries of K3 sigma models.

This talk:
It is not symmetries of nonperturbative spacetime BPS states of type IIA K3 compactifications.

Still leaves the possibility: Algebra of BPS states of the PERTURBATIVE BPS states of IIA on, say, K3 x S1.

# Part II

# Holography & Zamolochikov Volumes Of Moduli Spaces of Calabi-Yau Manifolds

Time permitting ……

G. Moore, ``Computation Of Some Zamolodchikov Volumes, With An Application,'' arXiv:1508.05612

# AdS3/CFT2

$$\mathcal{C}_M = \mathrm{Sym}^M(X)$$

$$X = K3, T4$$

The large M limit of these CFT's exist:

Holographically dual to IIB strings on

$$AdS_3 \times S^3 \times X$$

Rademacher series (and mock modular forms) are relevant to string theory.

[Dijkgraaf, Maldacena,Moore,Verlinde (2000)]

Some recent activity has centered on question:

``Do more general sequences $\{\mathcal{C}_M\}$ have holographic duals with weakly coupled gravity? "

This talk: CFT's are unitary and (4,4) supersymmetric:

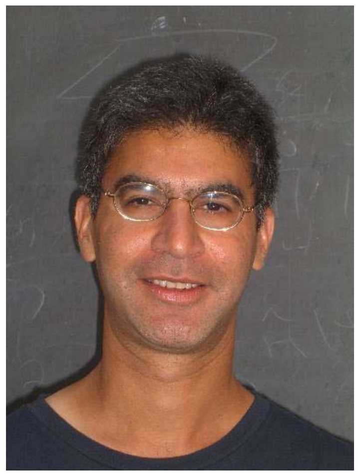
$$c = 6M$$

Put necessary conditions (e.g. existence of a Hawking-Page phase transition) on partition functions $Z(\mathcal{C}_M)$ for a holographic dual of an appropriate type to exist.

Keller; Hartman, Keller, Stoica; Haehl, Rangamani; Belin, Keller, Maloney; ….

Miranda Cheng

Nathan Benjamin



Shamit Kachru

Natalie Paquette

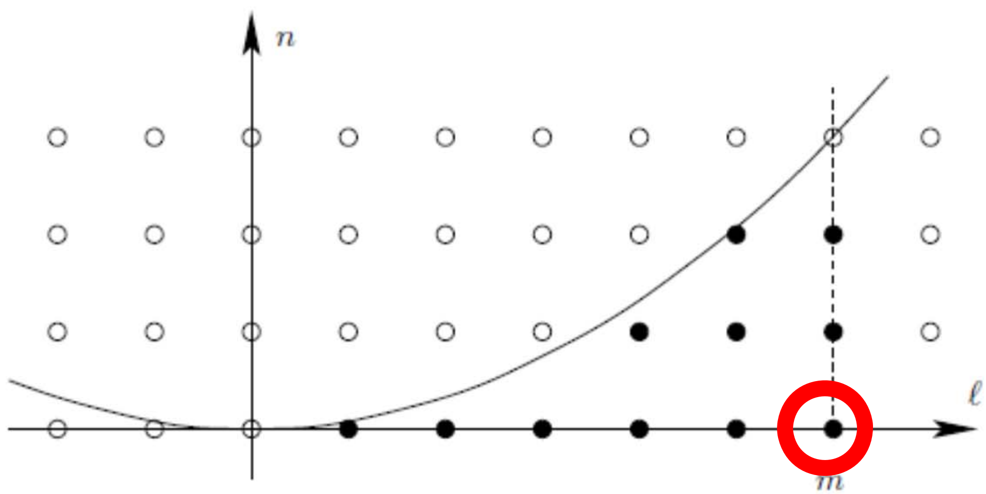
Our paper: Apply criterion of existence of a Hawking-Page phase transition to the elliptic genus.

# Reminder On Elliptic Genera

$$\mathcal{E}(\tau, z; \mathcal{C}) = \mathrm{Tr}_{\mathcal{H}_{RR}} q^{L_0 - c/24} e^{2\pi i z J_0} \bar{q}^{\bar{L}_0 - c/24} e^{i\pi (J_0 - \bar{J}_0)}$$

$$\mathcal{E}(\tau, z; \mathcal{C}) = \sum_{n,\ell \in \mathbb{Z}} c(n, \ell; \mathcal{C}) q^n y^\ell$$

Modular object: Weak Jacobi form of weight zero and index M.



# Extreme Polar Coefficient

$$\mathcal{E}(\tau, z; \mathcal{C}) = \mathfrak{e}(\mathcal{C})y^M + \cdots$$

Benjamin et. al. put constraints on coefficients of elliptic genera of a sequence $\{\mathcal{C}_M\}$ so that it exhibits HP transition. A corollary:

A ***necessary*** condition for $\{\mathcal{C}_M\}$ to exhibit a HP transition is that

$\mathfrak{e}(\mathcal{C}_M)$ has at most polynomial growth in M for $M \to \infty$

Just a necessary condition.

# Shamit's Question

``How *likely* is it for a sequence of CFT's $\{\ \mathcal{C}_M\ \}$ to have this HP phase transition? "

We'll now make that more precise, and give an answer.

# Zamolodchikov Metric

Space of CFT's is thought to have a topology. So we can speak of continuous families and connected components.

At smooth points the space is thought to be a manifold and there is a canonical isomorphism:

$$\Psi : V^{1,1}(\mathcal{C}) \to T_{\mathcal{C}}\mathcal{M}$$

$$\frac{\partial}{\partial t}|_0 S[t] = \int \mathcal{O} \qquad \Psi(\mathcal{O}) = \frac{\partial}{\partial t}|_0 = v \in T_{\mathcal{C}}\mathcal{M}$$

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\rangle := g_{\mathrm{Z}}(v,v)\frac{d^2 z_1 d^2 z_2}{|z_1 - z_2|^4}$$

# Strategy

Suppose we have an ensemble $\mathcal{E}$ of (4,4) CFTs:

$$\mathcal{E} = \amalg_M \mathcal{E}_M \qquad \mathcal{E}_M = \amalg_\alpha \mathcal{E}_{M,\alpha}$$

$$\sum_\alpha \mathrm{vol}_Z(\mathcal{E}_{M,\alpha}) < \infty$$

Then use the Z-measure to define a probability density on $\mathcal{E}_M$ for fixed M.

# Strategy – 2/2

Now suppose { $\mathcal{C}_M$ } is a sequence drawn from $\mathcal{E}$.

$$p_M(\kappa,\ell) := \sum_{\mathfrak{e}\leq \kappa M^\ell} \frac{\mathrm{vol}(\mathfrak{e};\mathcal{E}_M)}{\mathrm{vol}(\mathcal{E}_M)}$$

$$\wp(\ell) := \lim_{M\to\infty} p_M(\kappa,\ell)$$

$\wp(\ell)$ probability that a sequence drawn from $\mathcal{E}$ has extremal polar coefficient growing at most like a power $M^\ell$

# Multiplicative Ensembles

$\mathfrak{e}(\mathcal{C})$ is constant on each component: $\mathcal{C} \in \mathcal{E}_{M,\alpha}$

$$\mathfrak{e}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathfrak{e}(\mathcal{C}_1)\mathfrak{e}(\mathcal{C}_2)$$

Definition: A ***multiplicative ensemble*** satisfies:

$$\mathrm{vol}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{vol}(\mathcal{C}_1)\mathrm{vol}(\mathcal{C}_2)$$

Definition: A CFT $\mathcal{C}$ in a multiplicative ensemble is ***prime*** if it is not a product of CFT's (even up to deformation) each of which has m>0.

# A Generating Function

$\mathcal{C}_{m,\alpha}$ prime CFT's with c= 6m, $\alpha = 1, \dots, f_m$

$$\mathfrak{e}(m,\alpha) = \mathfrak{e}(\mathcal{C}_{m,\alpha})$$

$$v(m,\alpha) = \mathrm{vol}_Z(\mathcal{C}(m,\alpha))$$

$$\prod_{m=1}^{\infty} \prod_{\alpha=1}^{f_m} \frac{1}{1 - v(m,\alpha)\mathfrak{e}(m,\alpha)^{-s} q^m} = 1 + \sum_{M=1}^{\infty} \xi(s;M) q^M$$

$$\xi(s;M) = \sum_{\mathfrak{e}=1}^{\infty} \frac{\mathrm{vol}(\mathfrak{e};M)}{\mathfrak{e}^s}$$

# Some Representative (?) Ensembles

We do not know what the space of (4,4) CFT's is

We do not even know how to classify compact hyperkähler manifolds !

$$S^m K3 := \mathrm{Hilb}^m(K3)$$

$$S^m T4 := \mathrm{Hilb}^{m+1}(T4)/T4$$

$$\mathcal{E}_M = \{(S^1 X)^{n_1} \times \cdots \times (S^r X)^{n_r}\}$$

$$X = K3 \qquad X = T4$$

$$X \in \{K3, T4\}$$

$$M = n_1 + 2n_2 + \cdots + rn_r$$

# Moduli Spaces Of The Prime CFTs -1/2

These ensembles are multiplicative.

Primes: $S^m X$ Moduli space?

Moduli space for X

$$Q_{r,s} = II^{r+8s,r}$$

$$\mathcal{N}_{r+8s,r} = O_{\mathbb{Z}}(Q_{r,s}) \backslash O_{\mathbb{R}}(Q_{r,s}) / O(r+8s) \times O(r)$$

$$\mathcal{M}(X) = \begin{cases} \mathcal{N}_{4,4} & X = T4 \\ \mathcal{N}_{20,4} & X = K3 \end{cases}$$

# Moduli Spaces Of The Prime CFTs -2/2

One can derive $\mathcal{M}(S^m(X))$ using the attractor mechanism:

Dijkgraaf;
Seiberg & Witten

Begin with O(5,21) (or O(5,5)) moduli space of supergravity

Consider the subgroup fixing a primitive vector $u \in II^{r+8s,r}$

The conjugacy class only depends on $u^2 = 2m$

Then $\mathcal{M}(S^m(X))$ is:

$$O_{\mathbb{Z}}(Q_{r,s}, m) \backslash O_{\mathbb{R}}(Q_{r,s}, m) / O(r+8s) \times O(r-1)$$

$$(r,s) = \begin{cases} (5,0) & X = T4 \\ (5,2) & X = K3 \end{cases}$$

# A Generating Function

$\mathcal{C}_{m,\alpha}$ prime CFT's with c= 6m, $\alpha = 1, \ldots, f_m$

$$\prod_{m=1}^{\infty} \prod_{\alpha=1}^{f_m} \frac{1}{1 - v(m,\alpha)\mathfrak{e}(m,\alpha)^{-s} q^m} = 1 + \sum_{M=1}^{\infty} \xi(s;M) q^M$$

$$\xi(s;M) = \sum_{\mathfrak{e}=1}^{\infty} \frac{\mathrm{vol}(\mathfrak{e};M)}{\mathfrak{e}^s}$$


$$\mathfrak{e}(m,\alpha) = \mathfrak{e}(\mathcal{C}_{m,\alpha})$$

# Extreme Polar Coefficient

From the formula for the partition functions of symmetric product orbifolds we easily find

$$\mathfrak{e}(S^m K3) = m + 1$$

$$\mathfrak{e}(S^m T4) = m + 1$$

# A Generating Function

$\mathcal{C}_{m,\alpha}$ prime CFT's with c= 6m, $\alpha = 1, \ldots, f_m$

$$\prod_{m=1}^{\infty} \prod_{\alpha=1}^{f_m} \frac{1}{1 - v(m,\alpha)\mathfrak{e}(m,\alpha)^{-s} q^m} = 1 + \sum_{M=1}^{\infty} \xi(s;M) q^M$$

$$\xi(s;M) = \sum_{\mathfrak{e}=1}^{\infty} \frac{\mathrm{vol}(\mathfrak{e};M)}{\mathfrak{e}^s}$$


$$v(m,\alpha) = \mathrm{vol}_Z(\mathcal{C}(m,\alpha))$$

# Volumes

Using results from number theory, especially the ``mass formulae'' of Carl Ludwig Siegel, one can -- with some nontrivial work -- compute the Z-volumes of these spaces. For example:

$$\mathrm{vol}_Z(K3) =$$

$$\pi^{-40}\frac{(131)(283)(593)(617)(691)^2(3617)(43867)}{2^{40}\cdot 3^{34}\cdot 5^{15}\cdot 7^{9}\cdot 11^{5}\cdot 13^{4}\cdot 17^{3}\cdot 19^{3}\cdot 23}$$

$$\cong 1.66\times 10^{-61}$$

$$\mathrm{vol}_Z(S^m K3) = ????$$

Much harder, but from C.L. Siegel we get:

$$\frac{\mathrm{vol}(O_{\mathbb{Z}}(Q_{r,s},m)\backslash O_{\mathbb{R}}(Q_{r,s},m))}{\mathrm{vol}(O_{\mathbb{Z}}(Q_{r,s})\backslash O_{\mathbb{R}}(Q_{r,s}))} = \prod_{p<\infty} \alpha_p(m)$$

$$A(d,m,p^t) := \#\{v \mod p^t | Q_{r,s}(v) = 2m \mod p^t\}$$

$$\alpha_p(m) := \lim_{t\to\infty} \frac{A(d,m,p^t)}{p^{t(2d-1)}}$$

$$d = r + 4s$$

$$A(d,m,p^t) := \#\{v \mod p^t | Q_{r,s}(v) = 2m \mod p^t\}$$

$$A(d,p^e,p^t) := p^{t(2d-1)}(1-p^{-d})\frac{1-p^{-(e+1)(d-1)}}{1-p^{-(d-1)}}$$

For p an odd prime & t > e

$$A(d,2^e,2^t) := 2 \cdot 2^{t(2d-1)}(1-2^{-d})\frac{1-2^{-(e+1)(d-1)}}{1-2^{-(d-1)}}$$

$$\mathrm{vol}_Z(S^m K3) = \rho m^{42} f_{13}(m)$$

$$f_{13}(m) = \prod_{p|m} \frac{1 - p^{-12-12e_p(m)}}{1 - p^{-12}}$$

$$\rho = \pi^{-42} \frac{(103)(131)(283)(593)(617)(691)(3617)(43867)(2294797)}{2^{51} \cdot 3^{35} \cdot 5^{15} \cdot 7^{10} \cdot 11^{5} \cdot 13^{4} \cdot 17^{3} \cdot 19^{3} \cdot 23^{2}}$$

$$\cong 5.815 \times 10^{-63}$$

# Result For Probabilities

$$H_\ell(s) := \lim_{M\to\infty} (M+1)^{\ell s} \frac{\xi(s;M)}{\xi(0;M)}$$

$$= \lim_{M\to\infty} \sum_{\mathfrak{e}=M+1}^{2^M} \frac{\mathrm{vol}(\mathfrak{e};M)}{\mathrm{vol}(M)} \left(\frac{(M+1)^\ell}{\mathfrak{e}}\right)^s$$

$$\geq \lim_{M\to\infty} \kappa^{-s} p_M(\kappa,\ell) \geq 0$$

# Result For Probabilities

$$H_\ell(s) := \lim_{M\to\infty} (M+1)^{\ell s} \frac{\xi(s;M)}{\xi(0;M)}$$

Claim: The limit exists for all nonnegative $\ell$ and

$$H_\ell(s) = \begin{cases} 1 & s = 0 \\ 0 & s > 0 \end{cases}$$

$$\Rightarrow \wp(\ell) = 0$$

Conclusion: Almost every sequence $\{\mathcal{C}_M\}$ does not have a holographic dual

# Proof – 1/3

$$\prod_{m=1}^{\infty}\prod_{\alpha=1}^{f_m}\frac{1}{1-v(m,\alpha)\mathfrak{e}(m,\alpha)^{-s}q^m}=1+\sum_{M=1}^{\infty}\xi(s;M)q^M$$

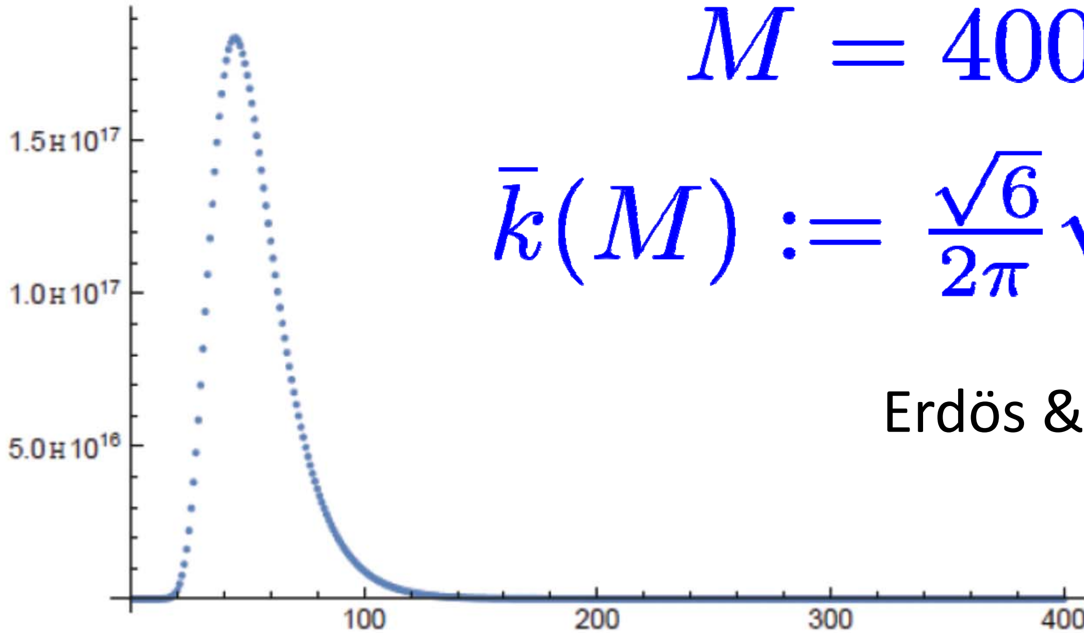
So $\xi$(s:M) is a sum over partitions:

$$M=\lambda_1+\cdots+\lambda_k$$

$$\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k$$

# Statistics Of Partitions

For large M the distribution of partitions into k parts is sharply peaked:




$$M = 400$$

$$\bar{k}(M) := \frac{\sqrt{6}}{2\pi}\sqrt{M}\log M$$

Erdös & Lehner

Moreover the ``typical'' partition has ``most'' parts of order: $\lambda_j \cong \sqrt{M}$

# Proof – 3/3

$\xi(s;M)$ is dominated by:

$$\xi(s;M) \cong V(\sqrt{M})^{\sqrt{M}}(\sqrt{M})^{-s\sqrt{M}}e^{2\pi\sqrt{\frac{M}{6}}}$$

$$H_\ell(s) := \lim_{M\to\infty}(M+1)^{\ell s}\frac{\xi(s;M)}{\xi(0;M)}$$

$$= \lim_{M\to\infty}(M+1)^{\ell s}(\sqrt{M})^{-s\sqrt{M}}$$

$$= \begin{cases} 1 & s=0 \\ 0 & s>0 \end{cases}$$

# Some Wild Speculation:

(Discussions with Shamit Kachru and Alex Maloney)

## ***Siegel Mass Formula:***

Two lattices $\Gamma_1$ and $\Gamma_2$ are in the same genus if

$$\Gamma_1 \oplus S \cong \Gamma_2 \oplus S$$

Even unimodular lattices of rank 8n form a single genus, and:

$$\sum_\alpha \frac{1}{|\mathrm{Aut}\Gamma_\alpha|} = \prod_p \alpha_p$$

# A Natural Ensemble & Measure

Consider the ensemble of ***holomorphic*** CFT's.

(What would they be dual to? Presumably some version of chiral gravity in 3d! )

Holomorphic CFTs have c = 24N

They are completely rigid

$$Z = \sum_\alpha \frac{1}{|\mathrm{Aut}(\mathcal{C}_\alpha)|} < \infty$$

$$\mu(\mathcal{C}_\alpha) = \frac{1}{Z} \frac{1}{|\mathrm{Aut}(\mathcal{C}_\alpha)|}$$

# Some Wild Speculation – 3/4

$$\mathcal{E}_N = \{\mathcal{C}(\Gamma, G) | G \subset \mathrm{Aut}(\Gamma) \quad \& \quad \Gamma \in II^{24N}\}$$

(Speculation: This set exhausts the set of c=24N holomorphic CFTs.)

Speculation: Using results on the mass formula for lattices with nontrivial automorphism we can again prove that sequences $\{\mathcal{C}_N\}$ with a holographic dual are measure zero.

# Even Wilder Speculation – 4/4

Define a ``genus'' to be an equivalence class under tensoring with a lattice theory of chiral scalar fields.

$$\sum_\alpha \frac{1}{|\mathrm{Aut}\mathcal{C}(\Gamma,G)_\alpha|} = \prod_p \alpha_p^{CFT}$$

Where the local densities are computed by counting automorphisms of the vertex operator algebra localized at a prime p.