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Notes For Talk At "Geometry and Physics ..." Conference, Miami, Jan. 2016

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Abstract: Notes for a talk at a conference honoring Ron Donagi's 60th birthday, Jan. 26, 2016. January 26, 2016

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1. Introductory Remarks

1.1 Preamble

It's a real pleasure to be speaking at this conference honoring Ron Donagi. I've always very much enjoyed discussing Physical Mathematics with him. My first really interesting discussions with Ron were about the relation between a certain class of supersymmetric black holes and K3 surfaces with complex multiplication, but there have been many others since then.

Also – as I'm sure many other people will mention - Ron deserves high praise for initiating the String-Math conferences: These have been a great success, filling in a real need.

So happy birthday Ron!

1.2 Today's Topic

My topic today concerns a mathematical way of formulating the "space of **BPS States**" in **d=4 N=2 field theories**.

If you like, it is a categorification of motivic DT invariants, at least in some special situations such as $\mathbb{P}^1$ families of resolved ADE singularities. I'm pretty sure that for some of the mathematicians present it will be a "new" formulation of BPS states. Moreover, the counting of BPS states will literally involve the computation of a character-valued index of a Dirac-type operator:

$$\mathcal{H}^{BPS} = \ker_{L^2} \not{D} \tag{1.1}$$

and the wall-crossing will literally be due to the failure of that Dirac-operator to be Fredholm. One of the main goals of the talk is to make this statement more intelligible and perhaps plausible.

But actually, it is not a new formulation - in fact, it is the oldest formulation there is, going back to the original paper of Witten and Olive in 1978. But it uses one of the physicist's secret weapons, possibly not generally appreciated by mathematicians that work on homological mirror symmetry. For the physicists here they will probably think the talk is pretty trivial, but I thought it might be useful for the mathematicians.

The secret weapon is the "semiclassical approximation." For d=4 N=2 field theories with a Lagrangian description we are talking about supersymmetric Yang-Mills theory, possibly coupled to matter multiplets, and semiclassically we can think in terms of BPS states as the groundstates of a particle moving in the moduli space of magnetic monopoles.

The considerable details were worked out in a project with **Andy Royston** at TAMU and **Dieter van den Bleeken**, at Bogazici U. in Turkey. Two papers were posted on the arXiv at the end of December: a long one, and a short summary that is meant to be readable. Since the long paper is long on details I thought I would balance that by giving a talk a bit short on details.

2. Monopole Moduli Space

We begin with a review of magnetic monopoles in YMH theory.

YMH means we choose a compact simple group G and a principal G-bundle $\mathcal{P} \to M_4$. The fields are a connection A on $\mathcal{P}$ and a section X of ad$\mathcal{P}$. The action is

$$\boxed{S = \frac{1}{g^2}\int_{M_4} (F, *F) + (DX, *DX)} \tag{2.1}$$

We can also add a theta angle and in the quantum theory the more relevant parameter is ♣Save on blackboard for modification later. ♣

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi \mathrm{i}}{g^2} \tag{2.2}$$

We're going to be talking about the limit

$$\mathrm{Im}\tau \to \infty \tag{2.3}$$

limit, and the leading approximation there is classical physics.

2.1 Smooth Monopoles

We are interesting in static smooth solutions on $\mathbb{R} \times \mathbb{R}^3$ with finite minimal energy subject to the boundary condition that on large spheres of sufficiently large radius

$$X : S^2_\infty \to \mathcal{O} \subset \mathfrak{g} \tag{2.4}$$

maps to an orbit of a regular element. Homotopy classes are in Λ_{cr} and in a suitable gauge (fixing a framing at a point at infinity):

$$\begin{aligned} X &\to X_\infty - \frac{\gamma_m}{2r} + \cdots \qquad\qquad r \to \infty \\ F &\to \frac{1}{2}\gamma_m \omega + \cdots \end{aligned} \tag{2.5}$$

where ω is a volume form on S^2 and γ_m is in the coroot lattice. Such solutions must satisfy the Bogomolnyi equation

$$F = *DX \tag{2.6}$$

on $\mathbb{R}^3$.

We are in fact interested in the moduli space $\mathcal{M}(\gamma_m; X_\infty)$ of such solutions. Let's discuss some properties:

1. X_∞ is regular so it selects a CSA $\mathfrak{t}$ and system of simple (co)roots: H_I, α_I. Then solutions exist iff γ_m is a strongly dominant coroot. Then $\mathcal{M}$ is a HK quotient and is a smooth HK manifold of quaternionic dimension

$$\langle \rho, \gamma_m \rangle \tag{2.7}$$

2. If we choose a splitting of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ then we choose a complex structure and in this complex structure the moduli space can also be viewed as the space of rational maps (Donaldson, Hurtubise, Jarvis)

$$Hol(\mathbb{P}^1 \to \mathcal{F} := G^c/B) \tag{2.8}$$

3. There are two triholomorphic groups of Killing symmetries that will be important to us: One is the an actin by T, via conjugation of the holomorphic map. The other (more obvious in from the symmetries of the Bogomolnyi equation) is induced by translations in space. In addition there is an $\mathfrak{so}(3)_{rot}$ Killing symmetry induced by rotation around some origin. It is not triholomorphic. Rather it rotates the complex structures.

4. The moduli space is a quotient

$$\mathcal{M} = \frac{\mathbb{R}^4 \times \mathcal{M}_0}{\mathbb{Z}} \tag{2.9}$$

and the metric product with $\mathbb{R}^4$ is obtained by looking at the orbits of the translations and the T action in the direction X_∞. $\mathcal{M}_0$ is called the **strongly centered** moduli space.

5. One of the main results of the talk is

    (Physics) Theorem: $H^{0,*}_{L^2}(\mathcal{M}_0, \bar\partial + \bar v)$, where $\bar v$ is the wedge with the (0,1) part of the T-action by $v \in \mathfrak{t}$, is primitive wrt a natural sl(2) action. (In particular, concentrated in the middle degree.

    This is just a teaser. More details later.

2.2 Singular Monopoles

We will also be interested in "singular monopoles." . Choose a point in R3, call it the origin and also demand

$$
\begin{aligned}
X &= -\frac{P}{2r} + \cdots \qquad\qquad r \to 0 \\
F &= \frac{1}{2} P\omega + \cdots
\end{aligned}
\tag{2.10}
$$

P describes the embedding of a Dirac monopole singularity in the nonabelian theory and is valued in the cocharacter lattice:

$$
P \in \Lambda_G \cong Hom(U(1), T). \tag{2.11}
$$

For singular monopoles γ_m is in the torsor $P + \Lambda_{cr}$.

Again we are interested in the moduli space (modulo gt's commuting with P at $r \to 0$) Denote it by $\underline{\mathcal{M}}(P, \gamma_m, X_\infty)$.

Analogous properties:

1. $\underline{\mathcal{M}}$ is HK with singularities. Conjecturally the space is nonempty iff the **relative magnetic charge** is **dominant**. The relative magnetic charge is defined by ♣What kind of singularities??? ♣

    $$
    \tilde\gamma := \gamma - P^- \tag{2.12}
    $$

    where where P^- is the Weyl image in the anti-fundamental chamber. When nonempty the moduli space has quaternionic dimension $\langle \rho, \tilde\gamma \rangle$.

2. There is a formulation in terms of rational maps and the map should have a singularity like z^P. ♣SHOULD SAY THIS BETTER!! ♣

3. There is a group $SO(3)_{rot} \times T$ of Killing symmetries where T is hyperholomorphic and $SO(3)$ is not.

4. There is an analogous statement about the L^2 $\bar\partial + \bar v$ cohomology of $\underline{\mathcal{M}}$ in any of its complex structures.

3. New Features From N=2 SUSY

We now embed the YMH theory into an N=2 SYM for gauge group G.

I will now make 10 loosely related remarks about the new ingredients and features we get with N=2 SUSY:

1. In terms of field content there is a second section of $\mathrm{ad}\mathcal{P}$ denoted Y. The bosonic action of the pure VM theory is:

$$\boxed{S = \frac{1}{g^2}\int_{M_4} (F,*F) + (DX,*DX) + (DY,*DY) + ([X,Y],[X,Y])} \tag{3.1}$$

plus a ϑ-term. Finite energy implies

$$Y_\infty \in \mathfrak{t} \tag{3.2}$$

2. We are also interested in "adding matter fields." That means there are fields which are sections of the bundle associated to $\mathcal{P}$ by a quaternionic representation $\mathcal{R}$ of G. The only new coupling constants are "mass parameters":

$$m \in \left(\mathrm{Sym}^2\mathcal{R}^\vee\right)^G \tag{3.3}$$

Altogether, an N=2 SYM theory is completely specified by a semisimple compact group G and a quaternionic representation $\mathcal{R}$ of G. The only parameters in the Lagrangian are gauge couplings τ_i for each simple factor G_i and mass parameters. It is "UV complete" if for each simple factor $C_2(\mathcal{R}_i) - 2h_i^\vee \leq 0$ where $\mathcal{R}_i$ is $\mathcal{R}$ considered as a representation of G_i.

♣Is this the right place? ♣

♣Is this relevant? Skip? ♣

3. Because of N=2 SUSY we have the whole apparatus of the "Coulomb branch of vacua" and "BPS states." The Coulomb branch means that there is a special Kähler manifold $\mathcal{B}$ and a local system of lattices with integral antisymmetric form

$$\Gamma \to \mathcal{B}^* \tag{3.4}$$

over $\mathcal{B}$ minus a "discriminant locus," equipped with a "central charge function"

$$Z : \Gamma \to \mathbb{C} \tag{3.5}$$

4. To put this into some context for mathematicians, for an important class of N=2 theories – sometimes called "theories of class S" – we can identify $\mathcal{B}$ as the base of a Hitchin system on some punctured Riemann surface C:

$$\mathcal{B} = \oplus_j H^0(C, K^{\otimes d_j} \otimes \mathcal{O}(D_j)) \tag{3.6}$$

and then the local system Γ is the fibration by H_1 of the Jacobian or Prym variety of the spectral curve:

$$\Sigma \subset T^*C \qquad\qquad \det(\lambda - \varphi) = 0 \tag{3.7}$$

where λ is the canonical Liouville 1-form. In the field theory context Σ is called the Seiberg-Witten curve and λ is called the Seiberg-Witten differential. The antisymmetric product on Γ is the intersection product of homology classes on Σ and the central charge function is defined by the periods of λ. *Indeed, a great paper by Donagi and Witten in 1995 was the first to point out this relation to Hitchin systems.* Ron also went on to write a very nice review of the relation to integrability as it was then known. From the class S viewpoint, the unifying principle is the six-dimensional $(2,0)$ superconformal theories, which can be compactified on Riemann surfaces C preserving 8 supersymmetries. For example, for pure SU(N) SYM with no matter the relevant Hitchin system is an $SU(N)$ system on $\mathbb{P}^1$ with two irregular singular points. So, in a suitable (holomorphic) gauge the Higgs field has the form:

$$\varphi \sim \left(\frac{e_{N1}}{z} + J_N + \mathcal{O}(z)\right)\frac{dz}{z} \qquad\qquad z \to 0 \tag{3.8}$$

5. For this talk one of the most important new features of N=2 SUSY is the existence of BPS states. The supersymmetry algebra implies a bound on the energy of quantum states in the Hilbert space on $\mathbb{R}^3$:

$$E \geq |Z| \tag{3.9}$$

and by definition BPS states are states that saturate the bound. We denote them by

$$\mathcal{H}^{BPS}(\gamma; u) \qquad\qquad \gamma \in \Gamma_u \qquad u \in \mathcal{B}^* \tag{3.10}$$

These spaces are representations of

$$\mathfrak{so}(3)_{rot} \oplus \mathfrak{su}(2)_R \tag{3.11}$$

This is part of the Lie algebra of the little group of a particle at rest: $\mathfrak{so}(3)_{rot}$ generates rotations around the particle and $\mathfrak{su}(2)_R$ is a global symmetry that rotates supersymmetry charges and is known in physics jargon as an "R-symmetry." In some contexts the character of this representation defines the "motivic DT invariants" and, as I said at the beginning, our goal is to give a formulation in terms of monopole moduli spaces.

6. Another feature of N=2 theories we will need are *supersymmetric line defects*. Quite generally, in QFT one can consider defects supported on submanifolds of spacetime of various dimensions. Local operators are supported at points. "Line defects" are supported on one-manifolds. A famous example in gauge theories is the "Wilson line" – it is just the trace of holonomy in a rep R:

$$\mathrm{Tr}_R Pexp \oint_\gamma A \tag{3.12}$$

In N=2 theories one can consider line defects that preserve supersymmetry, so for example the Wilson line is generalized to

$$\mathrm{Tr}_R Pexp \oint_\gamma (A + Y ds) \tag{3.13}$$

for a suitably normalized line element ds along γ. For supersymmetric line defects one always needs to choose a phase ζ which governs which supersymmetries that are preserved by the defect. That phase enters here through the relation

$$Y = \mathrm{Re}(\zeta^{-1}\varphi) \tag{3.14}$$

where φ is the complex adjoint-valued scalar field in the supersymmetric "vector-multiplet" with the gauge field. In general I'll denote a line defect with phase ζ as L_ζ.

7. The electromagnetic dual of a Wilson line is an 't Hooft line. In this talk we split space and time $\mathbb{R}_t \times \mathbb{R}^3$ and consider 't Hooft lines localized at a point $\vec{x}_0 \in \mathbb{R}^3$ and stretching along the time direction:

$$\gamma = \mathbb{R}_t \times \{\vec{x}_0\} \tag{3.15}$$

In this case an 't Hooft line defect is defined by specifying singular monopole boundary conditions on the fields around $\vec{x}_0$. So the defect depends on the data of the cocharacter P. We call these line defects $L_{P,\zeta}$.

8. In general, a susy line defect localized at a point $\vec{x}_0$ in space modifies the Hilbert space $\mathcal{H} \to \mathcal{H}_L$. For L_ζ the Bogomolnyi bound is modified to

$$E \geq -\mathrm{Re}(\zeta^{-1}Z) \tag{3.16}$$

Then "framed BPS states" are states in $\mathcal{H}_L$ saturating the Bogomolnyi bound. The charges of these states are valued in a torsor Γ_L for Γ. We denote the space of framed BPS states in the presence of L with charge $\gamma \in \Gamma_L$ at a point $u \in \mathcal{B}$ by

$$\overline{\underline{\mathcal{H}}}(L, \gamma; u) \tag{3.17}$$

Framed BPS are very useful in deriving wall-crossing formulae. For example my favorite derivation of the Kontsevich-Soibelman WCF for ordinary - vanilla - BPS states is obtained rather easily by demanding consistency of the a much more easily derived WCF for the framed BPS states.

Finally, two extra side remarks:

9. If a theory of class S is compactified on a circle of radius R, so spacetime is $S^1_R \times \mathbb{R}^3$ then the moduli space of vacua becomes not the base $\mathcal{B}$ but the entire Hitchin moduli space $\mathcal{M}^{Hitchin}$. It is a HK manifold with a HK metric that depends on R. Moreover, if a supersymmetric line defect of phase ζ is wrapped on the circle then the path integral $\langle L_\zeta \rangle$ is a holomorphic function on Hitchin moduli space in complex structure ζ, where ζ is on the equator of the twistor sphere, so that $\zeta = 0$ corresponds to the complex structure in which the Hitchin fibration is holomorphic. This remark will play a role at the end of the talk.

10. As a side remark, in an effort to make the monopoles I'm talking about a little more relevant to the mathematicians here, I should mention that the Hitchin moduli space for $SU(N)$ that I just mentioned on $C = \mathbb{P}^1 - \{0, \infty\}$ with two irregular singular points with the R-dependent HK metric is <u>also</u> a moduli space of periodic monopoles on $S^1_{1/R} \times \mathbb{R}^2$ for gauge group $SU(2)$ with magnetic charge N. So there is an isometric isomorphism of the form

$$\mathcal{M}^{Hitchin} \cong \mathcal{M}^{Monopole}(S^1_{1/R} \times \mathbb{R}^2, G = SU(2), \gamma_m = N) \tag{3.18}$$

This statement has a nice generalization to linear quiver gauge theories. This follows from results of Diaconescu, Hanany-Witten, and Cherkis-Kapustin. So as $R \to 0$ some Hitchin moduli spaces becomes the monopole moduli space I'm talking about.

4. Semiclassical Analysis

Now we want to consider the limit of the quantum theory as $g \to 0$, that is, as $\mathrm{Im}\tau \to \infty$.

This translates into taking a path $u_t \in \mathcal{B}^*$ to infinity.

Although walls of marginal stability do extend to infinity, along these paths there is eventually no wall-crossing and the space of BPS states stabilizes.

Moreover there is an (almost) canonical electric-magnetic duality splitting

$$\begin{aligned} \Gamma_g &= \Lambda_{cr} \oplus \Lambda_{wt} \\ \gamma &= \gamma_m \oplus \gamma^e \end{aligned} \tag{4.1}$$

Now in physics there is a procedure for systematically accounting for the quantum corrections to the classical theory in this region known as <u>**collective coordinates**</u> and the main upshot is that

$\mathcal{H}^{BPS}(\gamma; u)$ **is the space of supersymmetric groundstates of a supersymmetric particle moving on** $\mathcal{M}(\gamma_m; X_\infty)$**.**

$\overline{\underline{\mathcal{H}}}^{BPS}(L_{P,\zeta}, \gamma; u)$ **is the space of supersymmetric groundstates of a supersymmetric particle moving on** $\overline{\underline{\mathcal{M}}}(P, \gamma_m; X_\infty)$**.**

These statements look a bit odd: What is X_∞? and what about γ^e?

One at a time: A result of the collective coordinate analysis is that the Higgs fields are derived from ζ and a point $u \in \mathcal{B}$ via

$$\begin{aligned} X_\infty &= \mathrm{Im}(\zeta^{-1} a) \\ \mathcal{Y} = \frac{4\pi}{g^2} Y_\infty + \frac{\vartheta_0}{2\pi} X_\infty &= \mathrm{Im}(\zeta^{-1} a_D) \end{aligned} \tag{4.2}$$

where ζ is determined by the line defect or in the case of ordinary BPS states, ζ is the phase of the central charge, and $a, a_D \in \mathfrak{t}^c$ are standard coordinates of special Kähler geometry. In the Hitchin context

$$\begin{aligned} a^I &= \oint_{A^I} \lambda \\ a_{D,I} &= \oint_{B_I} \lambda \end{aligned} \tag{4.3}$$

Now, in the SQM for pure VM theory a quantum state is specified by a section of the spinor bundle over $\mathcal{M}$ or $\underline{\overline{\mathcal{M}}}$, and one of the four supersymmetry operators is a Dirac-like operator

$$D^{\mathcal{Y}} := D\!\!\!\!/ + \mathcal{Y} \tag{4.4}$$

where $D\!\!\!\!/$ is the ordinary where $\mathcal{Y}$ means Clifford multiplication with the 1-form dual to the vector field induced by the Lie algebra element $\mathcal{Y} \in \mathfrak{t}$.

When we include matter with quaternionic representation $\mathcal{R}$ the universal bundle induces a hyperholomorphic bundle $V_{\mathcal{R}}$ over $\mathcal{M}$ or $\underline{\overline{\mathcal{M}}}$ and now a quantum state is a section of $S \otimes V_{\mathcal{R}}$ and $D\!\!\!\!/$ is the corresponding Dirac operator.

Physical states should be L^2 and so we arrive at one of our main statements:

The space of framed BPS states of magnetic charge γ_m for $L_{P,\zeta}$ in a weak coupling chamber u is $\ker_{L^2} D^{\mathcal{Y}}$.

with a similar statement for the ordinary BPS states (skating over a subtlety here).

Now, what about the electric charge γ^e?

Well, the T action is hyperholomorphic. It lifts to the spinor bundle and commutes with the Dirac operator, so there is an isotypical decomposition:

$$\ker_{L^2} D^{\mathcal{Y}} = \oplus_{\gamma^e \in \Lambda_{wt}} (\ker_{L^2} D^{\mathcal{Y}})^{\gamma^e} \tag{4.5}$$

We finally arrive at:

$$\boxed{\underline{\overline{\mathcal{H}}}^{BPS}(L_{P,\zeta}, \gamma; u) = (\ker_{L^2} D^{\mathcal{Y}})^{\gamma^e}} \tag{4.6}$$

More is true: Recall that there is an $\mathfrak{so}(3)_{rot}$ Killing symmetry of $\mathcal{M}, \underline{\overline{\mathcal{M}}}$. Again it lifts to the spin bundle and while it does not commute with the Dirac operator it does preserve the kernel and of course it commutes with $\mathfrak{t}$ so each isotypical component $(\ker_{L^2} D^{\mathcal{Y}})^{\gamma^e}$ is a representation of $\mathfrak{so}(3)_{rot}$.

Moreover, if the moduli space has real dimension $4N$ then the commutant of the holonomy defines an $SU(2)$ group acting on the tangent bundle:

$$SU(2) \times Sp(N) \subset SO(4N) \tag{4.7}$$

Again the $SU(2)$ lifts to the spin bundle and again, while it does not commute with the Dirac operator it preserves the kernel so $(\ker_{L^2} D^{\mathcal{Y}})^{\gamma^e}$ is a representation of

$$\mathfrak{so}(3)_{rot} \oplus \mathfrak{su}(2) \tag{4.8}$$

Now, as I already mentioned, the space of BPS states is a representation of $\mathfrak{so}(3)_{rot} \oplus \mathfrak{su}(2)_R$. Our identification above is meant to be an isomorphism of representations.

5. Math Predictions

5.1 No-Exotics

Now we come to the math predictions.

The first is based on a "physics theorem" known as the no-exotics theorem. It says - basically - that the the $SU(2)$ "R-symmetry" acts trivially on the space of BPS states.

More precisely, for the framed case it acts trivially and in the case of vanilla BPS states we have

$$\mathcal{H}^{BPS}(\gamma; u) \cong [(\mathbf{2}; \mathbf{1}) \oplus (\mathbf{1}; \mathbf{2})] \otimes \mathcal{H}_0^{BPS}(\gamma; u) \tag{5.1}$$

In this case the no-exotics theorem says that $SU(2)_R$ acts trivially on $\mathcal{H}_0^{BPS}(\gamma; u)$.

The theorem was proven for pure SU(N) SYM by

C(huang), D(iaconescu),M(anschot),M(oore),S(oibelman)

using the motivic structure of the moduli space of representations of a certain quiver, and this was extended to ADE groups by

del Zotto and Sen

But recently a proof of much greater applicability has been found by

Cordova and Dumitrescu

Here is the geometrical meaning:

First the above factorization of BPS spaces reflects the factorization of the moduli space

$$\mathcal{M} = \frac{\mathbb{R}^4 \times \mathcal{M}_0}{\mathbb{Z}} \tag{5.2}$$

So, actually, the proper formulation of vanilla BPS states is actually the L^2-kernel of Dirac on the strongly centered moduli space $\mathcal{M}_0$ that is suitably equivariant under the action of the Deck group $\mathbb{Z}$. (The details of the latter are a bit subtle.)

$$\boxed{\mathcal{H}_0^{BPS}(\gamma; u) = (\ker_{L^2} D_0^{\mathcal{Y}})^{\gamma^e, \mathbb{Z}}} \tag{5.3}$$

Second: $-1 \in SU(2)$ acts on the spin bundle as the Clifford volume element in the orientation given by any of the complex structures. Therefore, the L^2-kernel is purely chiral and not anti-chiral. The dimension is the index.

Third: If we choose a complex structure then

$$S \cong \Lambda^{0,*} \otimes K^{1/2} \tag{5.4}$$

with $K^{1/2}$ trivializable and a combination of supersymmetry operators becomes

$$\bar{\partial} + \mathcal{Y}^{0,1} \tag{5.5}$$

Moreover the generators of the $\mathfrak{su}(2)_R$ have a kind of Lefshetz form:

$$\begin{aligned} I^3|_{\Omega^{0,q}} &= \frac{1}{2}(q - N)\mathbf{1} \\ I^+ = \omega^{0,2}\wedge \qquad & \qquad I^- = \iota(\omega^{2,0}) \end{aligned} \tag{5.6}$$

so the statement of the no-exotics theorem says that the L^2-cohomology of the twisted $\bar{\partial}$ operator is primitive. This is the basis for the teaser made towards the beginning of the talk.

5.2 Exact Spectrum

Once we have a BPS spectrum in one chamber of the Coulomb branch we can – in principle – find it in every other chamber by wall-crossing. One nice thing is that in the chamber with $\mathcal{Y}$ we know the spectrum of vanilla BPS states immediately! Then the Dirac operator becomes the ordinary Dirac operator, but then it is easy to see there are no L^2 harmonic spinors on $\mathcal{M}_0$, unless $\mathcal{M}_0$ is a point. This is only the case when γ_m is a simple coroot, and then quantization of the remaining $\mathbb{R}^3 \times S^1$ factor is straightforward so we get the BPS spectrum:

1. VM's $\gamma = 0 \oplus \alpha$, $\alpha \in \Delta(\mathfrak{g})$
2. HM's $\gamma = \pm H_I \oplus n\alpha_I$, $n \in \mathbb{Z}$.

For G= SU(N) this was also shown using quiver techniques by Chuang et. al.

5.3 Wall-Crossing

It is not difficult to translate the usual criteria for wall-crossing into this semiclassical language. One finds a linear condition on X_∞, Y_∞ of the form

$$(\gamma_m, \mathcal{Y}) + \langle \gamma^e, X \rangle = 0, \tag{5.7}$$

for suitable electromagnetic charges $\gamma_m \oplus \gamma^e$, at which the Dirac operators $D^{\mathcal{Y}}$ must fail to be Fredholm.

The details are in our papers.

It would be nice to give a direct mathematical explanation of why they fail to be Fredholm.

6. Deriving An Index Theorem?

Finally, I would like to sketch something I'm trying to figure out right now.

Let us focus on theories of class S: So it is some N=2 SYM associated to a Hitchin system for G_S on a punctured Riemann surface C with some boundary conditions at punctures. (In general, G_S is a compact Lie group different from the four-dimensional gauge group G.)

I mentioned that on $S^1 \times \mathbb{R}^3$ with the line defects wrapped around the circle at a definite point in $\mathbb{R}^3$ the path integral $\langle L_\zeta \rangle$ for ζ-supersymmetric line defects becomes a holomorphic function on the Hitchin moduli space in complex structure ζ. As a holomorphic symplectic manifold this is isomorphic to the moduli space of G_S^c local systems with prescribed monodromy at the punctures.

In that context, a typical supersymmetric line defect is defined by a choice of G_S-rep R and an isotopy class of a path $\wp \subset C$. Then

$$\langle L_{R,\wp,\zeta} \rangle = \mathrm{Tr}_R \mathrm{Hol}(\wp) \tag{6.1}$$

Now in some work with Gaiotto and Neitzke we argued that there should be an exact expansion for these functions in terms of the BPS index for framed BPS states:

$$\langle L_\zeta \rangle = \sum_{\gamma \in \Gamma_L} \underline{\overline{\Omega}}(L, \gamma) \mathcal{Y}_\gamma \tag{6.2}$$

where $\mathcal{Y}_\gamma$ are a set of holomorphic functions on $\mathcal{M}_\zeta$ satisfying

$$\mathcal{Y}_{\gamma_1} \mathcal{Y}_{\gamma_2} = (-1)^{\langle\langle \gamma_1, \gamma_2 \rangle\rangle} \mathcal{Y}_{\gamma_1 + \gamma_2} \tag{6.3}$$

and satisfying some integral TBA-like equations. If we choose a basis for Γ then the $\mathcal{Y}_{\gamma_i}$ are Darboux coordinates for the holomorphic symplectic manifold $\mathcal{M}_\zeta$, closely related to "shear" or "Fock-Goncharov" coordinates. They are patchwise defined and related across patches by cluster-like transformations.

On the other hand, Ito-Okuda-Taki performed a localization computation of the path integral for a certain class of theories (including the $N = 2$ theories studied by Donagi and Witten) and for 't Hooft lines they found an expression of the form

$$\langle L_{P,\zeta} \rangle = \sum_{\gamma_m \in P + \Lambda_{cr}} e^{2\pi \mathrm{i}(\mathfrak{b}, \gamma_m)} Z_{\gamma_m}(\mathfrak{a}) \tag{6.4}$$

where $(\mathfrak{a}, \mathfrak{b})$ are complexified Fenchel-Nielson-type coordinates on the moduli space of local systems: The $\mathfrak{a}$ describe the conjugacy classes of holonomies around a system of cutting curves for a pants decomposition of C and $\mathfrak{b}$ is a dual set of coordinates.

Okuda et. al. furthermore claim that $Z_{\gamma_m}(\mathfrak{a})$ is related to an integral of certain characteristic classes over moduli spaces of monopoles. (This comes directly out of the localization computation.)

Now, in general, in patches where we can compare these coordinates the transformation of variables is very complicated.

However, in a weak-coupling chamber, there should be an asymptotic change of coordinates that is simple enough that we can compare the two expressions. The net result seems to lead to a formula like

$$\mathrm{Tr}_{\ker_{L^2}(D^{\mathcal{Y}})} t(-y)^{2J_3} = \int_{\underline{\mathcal{M}}} \mu \tag{6.5}$$

where μ is constructed from equivariant characteristic classes. So we get a kind of L^2-index theorem in this context. Many details remain to be clarified here.

7. Conclusions

So, to conclude, the oldest formulation of BPS states can be combined with recent insights to yield some interesting results. I hope you found the talk amusing. Thank you.