Partition Functions Of Twisted Supersymmetric Gauge Theories On Four-Manifolds Via u-Plane Integrals

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Mostly review; new work with I. Nidaiev;
work in progress with Jan Manschot.

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Donaldson Invariants Of 4-folds

X: Smooth, compact, oriented, $\partial X = \emptyset$, $(\pi_1(X) = 0)$

$P \to X$: Principal SO(3) bundle.

X has metric $g_{\mu\nu}$. Consider moduli space of instantons:

$$\mathcal{M} := \{ A: F + * F = 0\} \, mod \, \mathcal{G}$$

Donaldson defines cohomology classes in $\mathcal{M}$ associated to points and surfaces in X : $\mu(p)$ & $\mu(S)$

$$\wp_D(p^\ell S^r) := \int_{\mathcal{M}} \mu(p)^\ell \, \mu(S)^r$$

Independent of metric! $\Rightarrow$ smooth invariants of X.

Combined with Freedman theorem: Spectacular!

Witten’s Interpretation: Topologically Twisted SYM On X

Consider N=2 SYM theory on X for gauge group G

Witten’s ``topological twisting’’: Couple to special external gauge fields for certain global symmetries.

Result: Fermion fields and susy operators are differential forms; The twisted theory is defined on non-spin manifolds.

And there is a scalar susy operator with $Q^2 = 0$

Formally: Correlation functions of operators in Q-coho. localize to integrals over the moduli spaces of G-ASD connections (or generalizations thereof).

Witten’s proposal: For $G = SU(2)$ correlation functions of the special operators are the Donaldson polynomials.

Local Observables

$$\wp \in Inv(\mathfrak{g}) \Rightarrow \quad U = \wp(\phi) \qquad \phi \in \Omega^0(adP \otimes \mathbb{C})$$

$$\begin{tikzcd}
H_*(X,\mathbb{C}) \arrow[rr, "\text{Descent formalism}"] \arrow[dr, "\mu"'] & & H^*(Fieldspace; Q) \arrow[dl, "\text{Localization identity}"] \\
& H^*(\mathcal{M}) &
\end{tikzcd}$$

$$\mathfrak{g} = \mathfrak{su}(2) \qquad U(p) = Tr_2\left(\frac{\phi^2}{8\pi^2}\right) \qquad U(S) \sim \int_S Tr(\phi F + \psi^2)$$

Donaldson-Witten Partition Function

$$Z_{DW}(p,s) = \langle e^{U(p)+U(S)} \rangle_\Lambda$$

$$\int_X Tr\, F^2 = 8\,\pi^2 k$$

$$= \sum_k \Lambda^{\frac{1}{2}\dim(\mathcal{M}_k)} \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)}$$

Strategy: Evaluate in LEET $\Rightarrow$ Witten (1994) introduces the Seiberg-Witten invariants.

Major success in
Physical Mathematics.

What About Other N=2 Theories?

Natural Question: Given the successful application of $\mathcal{N} = 2$ SYM for SU(2) to the theory of 4-manifold invariants, are there interesting applications of OTHER $\mathcal{N} = 2$ field theories?

Topological twisting just depends on $SU(2)_R$ symmetry and makes sense for any $\mathcal{N} = 2$ theory.

$$Z^{\mathcal{T}} := \langle e^{U(p)+U(S)} \rangle_{\mathcal{T}}$$

Also an interesting exercise in QFT to compute correlation functions of nontrivial theories in 4d.

SU(2) With Matter

$\mathcal{R} = 2^{\oplus N_{fl}}$ Mass parameters $m_f \in \mathbb{C},\ f = 1, \ldots, N_{fl}$

$M \in \Gamma(S^+ \otimes E^{\oplus N_{fl}}) \Rightarrow \mathrm{w}_2(\mathrm{P}) = w_2(X)$

$\mathcal{N} = 2^*: \mathcal{R} = (adj_{\mathbb{C}} \oplus adj_{\mathbb{C}}^*) \quad m \in \mathbb{C}$

$M \in \Gamma(S^+ \otimes L^{\frac{1}{2}} \otimes adP) \Rightarrow w_2(L) = w_2(P)$

[Labastida-Marino ‘98]

UV Interpretation

$$Z(p,S) = \langle e^{U(p)+U(S)} \rangle_{\mathcal{T}}$$

$$= \sum_k \Lambda^{\frac{1}{2}\dim(\mathcal{M}_k)} \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)}$$

But now $\mathcal{M}_k$:
is the moduli space of:

$$F^+ = \mathcal{D}(M, \bar{M}) \qquad \gamma \cdot D\, M = 0$$

U(1) case: Seiberg-Witten equations.

``Generalized monopole equations''

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

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Coulomb Branch Vacua On $\mathbb{R}^4$

$SU(2) \to U(1)$ by vev of adjoint Higgs field ϕ:

Order parameter: $u = \langle U(\phi) \rangle \in \mathbb{C}$

Coulomb branch: $\mathcal{B} = \mathfrak{t} \otimes \mathbb{C}/W \cong \mathbb{C}$ $adP \to L^2 \oplus \mathcal{O} \oplus L^{-2}$

Photon: Connection A on L $U(1)\ VM$: (a, A, χ, ψ, η)

a: complex scalar field on $\mathbb{R}^4$:

Compute couplings in U(1) LEET.
Then compute path integral with this action.
Then integrate over vacua.

Seiberg-Witten Theory: 1/2

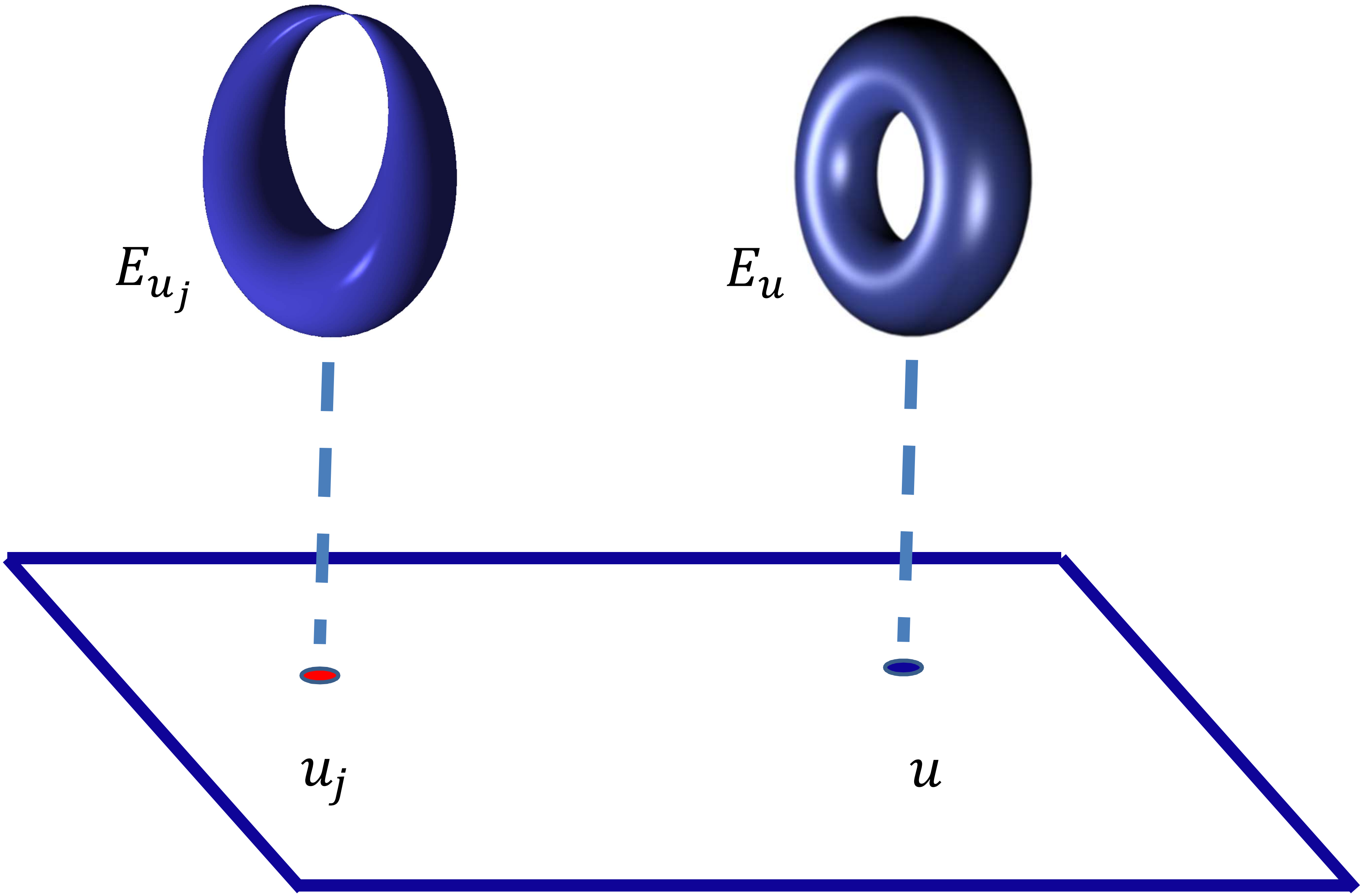
For G=SU(2) SYM coupled to matter the LEET can be deduced from a holomorphic family of elliptic curves with differential:

E_u: $y^2 = 4x^3 - g_2 x - g_3$ $\quad \frac{d\lambda}{du} = \frac{dx}{y} \quad u \in \mathbb{C}$

g_2, g_3 are polynomials in u, masses, Λ, modular functions of τ_0

$\Delta := 4(g_2^3 - 27\, g_3^2)$: polynomial in u

$\Delta(u_j) = 0$: Discriminant locus


E_{u_j}
E_u
u_j
u

Examples

$$N_{fl} = 0: \quad y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2)$$

$$\mathcal{N} = 2^*: y^2 = \prod_{i=1}^{3} (x - \alpha_i) \qquad \alpha_i = u\, e_i(\tau_0) + \frac{m^2}{4} e_i(\tau_0)^2$$

$$N_{fl} = 4: \quad y^2 = W_1 W_2 W_3 + \eta^{12} \sum_{i=1}^{3} R_4^i W_i + \eta^{24}\, R_6$$

$$W_i = x - u\, e_i(\tau_0) - R_2 e_i(\tau_0)^2$$

$$R_2 \sim Tr_{8_i} \mathfrak{m}^2 \qquad R_4^i \sim Tr_{8_i} \mathfrak{m}^4 \quad R_6 \sim Tr_{8_i} \mathfrak{m}^6$$

Δ: 6th order polynomial in u with $\sim 2 \times 10^3$ terms

Local System Of Charges

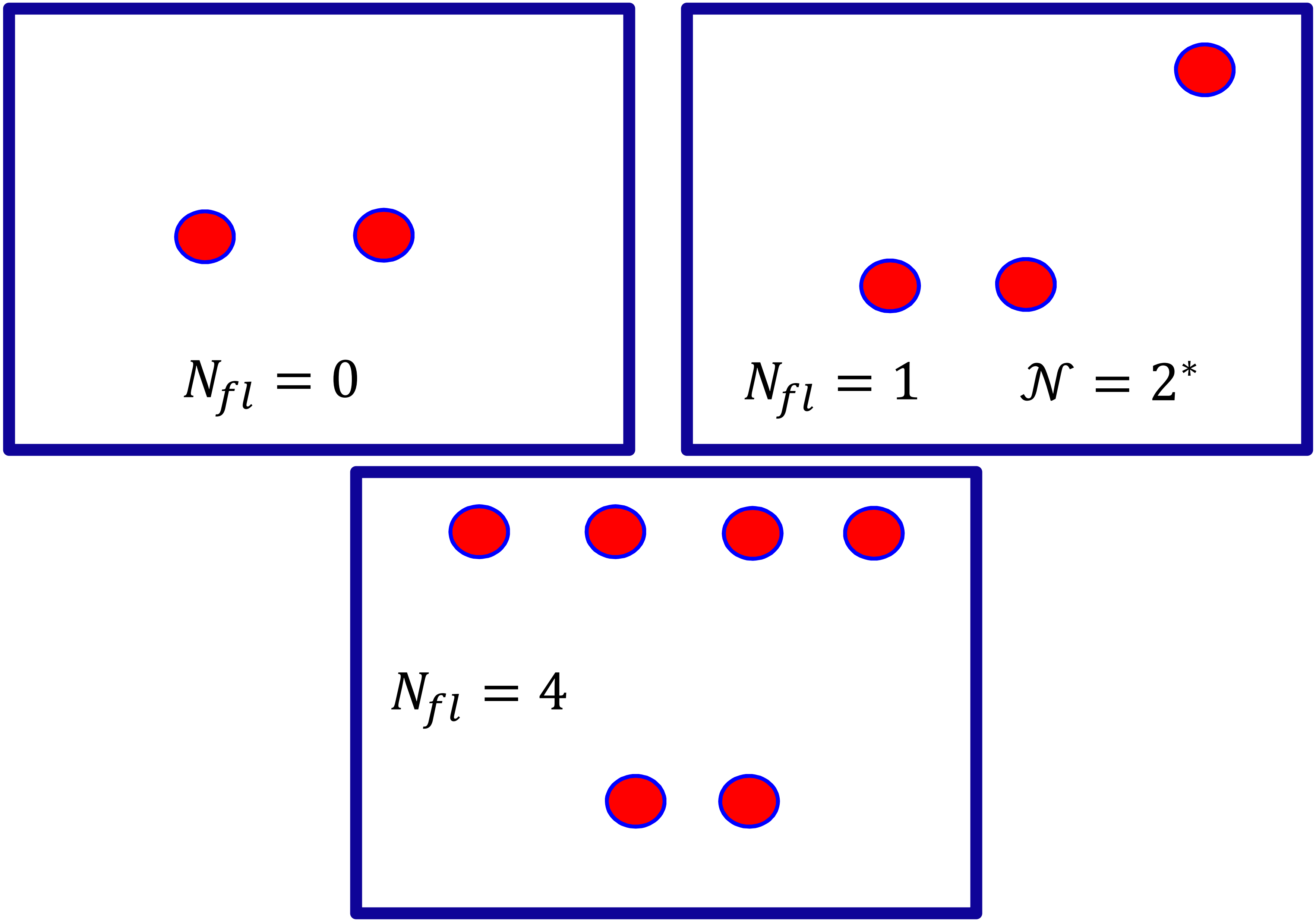
Electro-mag. charge lattice: $\Gamma_u = H_1(E_u;\mathbb{Z})$

has nontrivial monodromy around discriminant locus: $\Delta(u_j) = 0$

LEET: Requires choosing a duality frame:

$$\Gamma_u \cong \mathbb{Z}\gamma_e \oplus \mathbb{Z}\gamma_m \Rightarrow \tau(u)$$

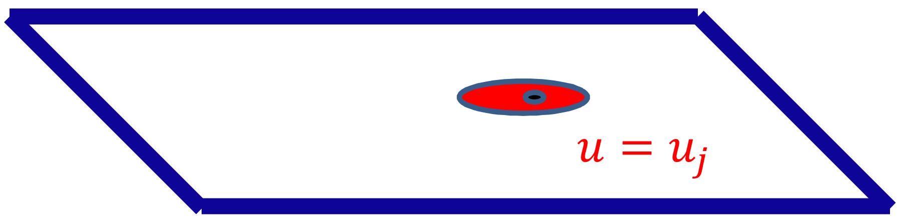
$$S \sim \int_X \bar{\tau} F_+^2 + \tau F_-^2 + \cdots$$



LEET breaks down at $u = u_j$ where $Im(\tau) \to 0$

Seiberg-Witten Theory: 2/2

LEET breaks down because there are new massless fields associated to BPS states



$$U(1)_j \; VM: \quad (a, A, \chi, \psi, \eta)_j$$

LEET $\mathrm{u} \in \mathcal{U}_j$:

$$+$$

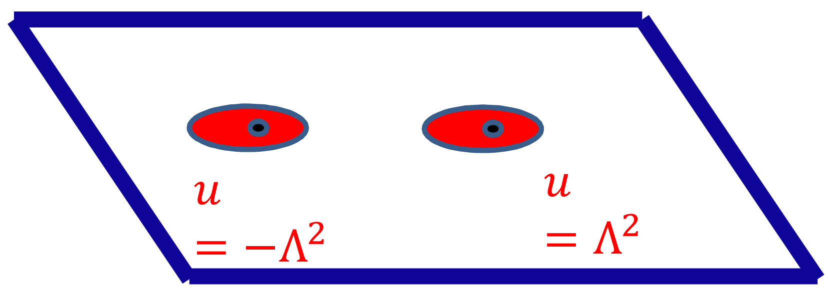
Charge 1 HM: $(M = q \oplus \tilde{q}^*, \cdots)$

Evaluate Z_{DW} Using LEET

$$Z_{DW}(p,S) = \langle e^{U(p)+U(S)} \rangle_\Lambda$$

$$= \sum_k \Lambda_0^{\frac{1}{2}\dim(\mathcal{M}_k)} \int_{\mathcal{M}_k} e^{\mu(p)+\mu(S)}$$

$$\boxed{Z_{DW} = Z_u + \sum_{u_j} Z_j^{SW}}$$



u-Plane Integral Z_u

Can be computed explicitly from QFT of LEET

Vanishes if $b_2^+ > 1$.   When $b_2^+ = 1$:

$$Z_u = \int du \, d\bar{u} \; \mathcal{H} \; \Psi$$

$\mathcal{H}$ is ***holomorphic*** and ***metric-independent***

Ψ: Sum over line bundles for the U(1) photon.

(Remnant of sum over SU(2) gauge bundles.)

$$\Psi \sim \sum_{\lambda = c_1(L)} e^{-i\,\pi\bar{\tau}(\bar{u})\lambda_+^2 - i\,\pi\tau(u)\lambda_-^2}$$

NOT holomorphic and ***metric- DEPENDENT***

Initial Comments On Z_u

Z_u is a very subtle integral.

It requires careful regularization and definition.

(It is also related to integrals from number theory such as ``Θ –lifts'' and mock modular forms…)

But first let's finish writing down the full answer for the partition function.

Contributions From $\mathcal{U}_j$

Path integral for $U(1)_j$ VM + HM:

General considerations imply $Z_j^{SW} =$

$$\sum_{\lambda \in \frac{1}{2} w_2(X) + H^2(X,\mathbb{Z})} SW(\lambda) e^{2\pi i \, \lambda\cdot\lambda_0} \, R_\lambda(p,S)$$

Special coordinate: a_j $\qquad u = u_j + \kappa_j a_j + \mathcal{O}(a_j^2)$

$$R_\lambda(p,S) = Res\left[\left(\frac{da_j}{a_j^{1+\frac{d(\lambda)}{2}}} \right) e^{2pu + S^2 T(u) + i\left(\frac{du}{da_D}\right) S\cdot\lambda} C(u)^{\lambda^2} P(u)^{\sigma} E(u)^{\chi} \right]$$

C,*P*,*E*:Universal functions. In principle computable.

$$d(\lambda) = \frac{(2\lambda)^2 - c^2}{4} \qquad c^2 = 2\chi + 3\,\sigma$$

Deriving C,P,E From Wall-Crossing

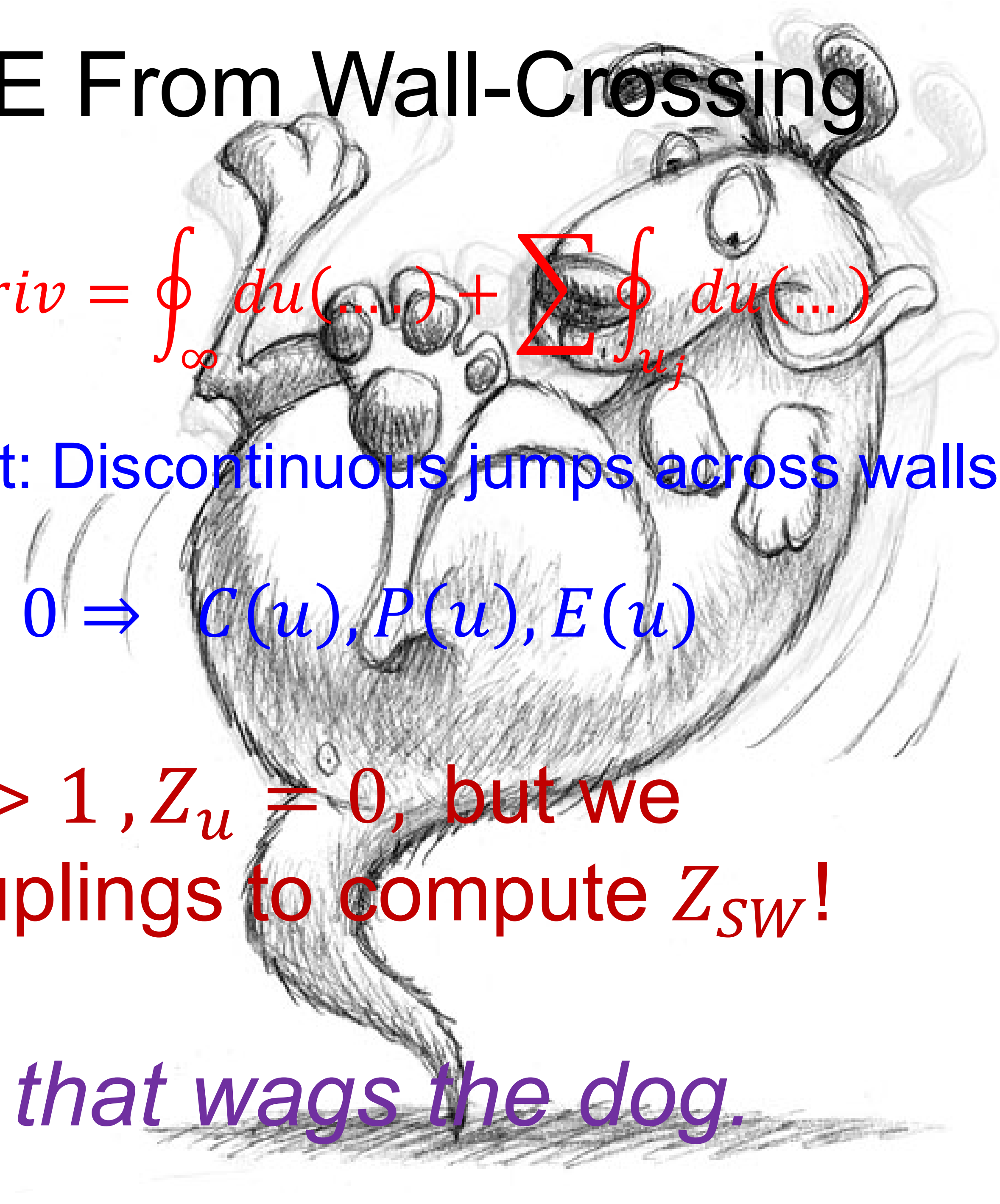
$$\frac{d}{dg_{\mu\nu}} Z_u = \int Tot\ deriv = \oint_{\infty} du(\dots) + \sum \oint_{u_j} du(\dots)$$

Z_u piecewise constant: Discontinuous jumps across walls:

$$\Delta_j Z_u + \Delta Z_j^{SW} = 0 \Rightarrow \quad C(u), P(u), E(u)$$

Then for $b_2^+ > 1$, $Z_u = 0$, but we know the couplings to compute Z_{SW}!

Z_u is the tail that wags the dog.



Witten Conjecture

$\Rightarrow$ a formula for all X with ${b_2}^+ > 0$. For $b_2^+(X) > 1$ we derive ``Witten's conjecture'':

$$Z_{DW}^{\xi}(p,s) = 2^{c^2-\chi_h}\left(e^{\frac{1}{2}S^2+2p} \sum_{\lambda} SW(\lambda) e^{2\pi i\, \lambda\cdot\lambda_0} e^{2S\cdot\lambda} + e^{-\frac{1}{2}S^2-2p} \sum_{\lambda} SW(\lambda) e^{2\pi\ \lambda\cdot\lambda_0} e^{-2iS\cdot\lambda}\right)$$

$$\chi_h = \frac{\chi+\sigma}{4} \qquad c^2 = 2\chi + 3\,\sigma$$

Example: $X = K3, w_2(P) = 0$:

$$Z = \sinh\left(\frac{S^2}{2} + 2\,p\right)$$

Generalization To $N_{fl} > 0$

$b_2^+ > 1$ $\quad$ $$Z(p;S;m_f) = \sum_{j=1}^{2+N_{fl}} Z(p,S;m_f;u_j)$$

$$Z(p,S;m_f;u_j) = \alpha^\chi \beta^\sigma \sum_\lambda SW(\lambda) e^{2\pi i\, \lambda\cdot\lambda_0} R_j(p,S)$$

X is SWST $\Rightarrow$

$R_j(p,S)$ is computable explicitly as a function of $p, S, m_f, \Lambda, \tau_0$ from first order degeneration of the SW curve.

$$R_j(p,s) = \mu_j^{\chi_h} \left(\frac{du}{da}\right)^{\chi_h+\sigma} \exp\left(2p\, u_j + S^2 T(u_j) - i\, \left(\frac{du}{da}\right)_j S\cdot\lambda \right)$$

$$u = u_j + \mu_j q_j + \mathcal{O}(q_j^2)$$

$$y^2 = 4x^3 - g_2(u,m)x - g_3(u,m)$$

Assume a simple zero for Δ as $u \to u_j$

Choose local duality frame with $a_j \to 0$

Nonvanishing period: $\frac{da_j}{du}$ and $\frac{da_{j,D}}{du} \to i\,\infty$:

$$\left(\frac{da_j}{du}\right)^2 \Big|_{u_j} \sim \left(\frac{g_2}{g_3}\right)\Big|_{u_j} \qquad \mu_j \sim \left(\frac{g_2^3}{\Delta'}\right)\Big|_{u_j}$$

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Application 1: Case Of $\mathcal{N} = 2^*$

Using these methods analogous formulae were worked out for N=2*, by Labastida-Lozano in 1998, but only in the case when X is spin.

They checked S-duality for the case $b_2^+ > 1$

The generalization to X which is NOT spin is nontrivial and involves new effective interactions not in the literature.

$$(u - u_1)^{-\frac{c_1(\mathfrak{s})^2}{8}} e^{-i\frac{\partial a_D}{\partial m}\lambda \cdot c_1(\mathfrak{s})}$$

Application 2: S-Duality Of $N_{fl} = 4$

$Z(p, S; \tau_0)$ is expected to have modular properties:

$$\sum_{j=1}^{6} \mu_j^{\chi_h} \kappa_j^{\chi_h+\sigma} e^{p\, u_j + i\, \kappa_j \lambda \cdot S + \cdots}$$

$$\kappa_j = \left(\frac{g_3(u_j)}{g_2(u_j)}\right)^{\frac{1}{2}}$$

$$\mu_j = \eta^{-24}(\tau_0) g_2(u_j)^3 \prod_k (u_j - u_k)^{-1}$$

Sum over j gives ***symmetric*** function of u_j $\Rightarrow$

Z will be modular:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$Z\left(\frac{p + \frac{c}{c\tau_0 + d} S^2}{(c\,\tau_0 + d)^2}, \frac{S}{(c\tau_0 + d)^2}; \frac{a\tau_0 + b}{c\tau_0 + d}; \gamma \cdot \mathfrak{m}\right)$$

$$= (c\tau_0 + d)^{\chi + 3\sigma}\, Z(p, S; \tau_0; \mathfrak{m})$$

(Neglecting α, β coefficients.)

Application 3: AD3 Partition Function

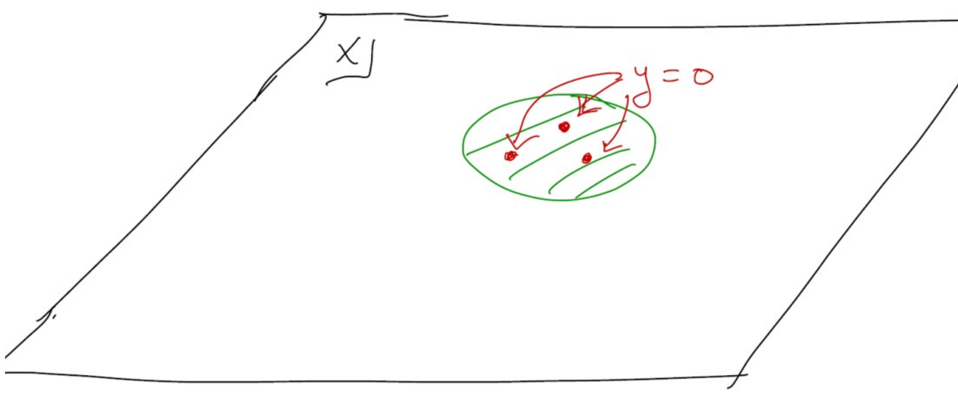
Consider $N_{fl} = 1$. At a critical point $m = m_*$ two singularities $u_\pm$ collide at $u = u_*$ and the SW curve becomes a cusp: $y^2 = x^3$ [Argyres,Plesser,Seiberg,Witten]

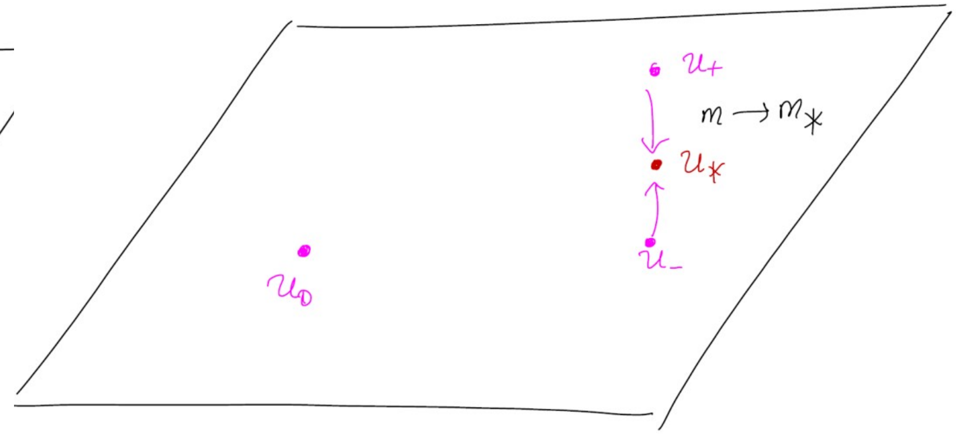
Two mutually nonlocal BPS states have vanishing mass:

$$\oint_{\gamma_1} \lambda \to 0 \qquad \oint_{\gamma_2} \lambda \to 0 \qquad \gamma_1 \cdot \gamma_2 \neq 0$$

Physically: No local Lagrangian for the LEET : Signals a nontrivial superconformal field theory appears in the IR in the limit $m \to m_*$

AD3 From $SU(2)\ N_f = 1$





SW curve in the scaling region:

$$y^2 = x^3 - 3\,\Lambda_{AD}^2\, x + u_{AD}$$

$$\Lambda_{AD}^2 \sim (m - m_*)$$

Call it the ``AD3-Family over the u_{AD} −plane"

AD3 Partition Function - 1

Work with Iurii Nidaiev

The $SU(2)\ N_{fl} = 1$ u-plane integral has a nontrivial contribution from the scaling region $u_{\pm} \to u_{*}$

$$\lim_{m \to m_*} Z_u \; - \; \int du\, d\bar{u} \lim_{m \to m_*} Measure(u, \bar{u}; m) \; \neq 0$$

Limit and integration commute except in an infinitesimal region around u_*

Attribute the discrepancy to the contribution of the AD3 theory

AD3 Partition Function - 2

1. Limit $m \to m_*$ exists. (No noncompact Higgs branch.)

2. The partition function is a sum over all Q-invariant field configurations.

3. Scaling region near u_* governed by AD3 theory.

$\lim_{m \to m_*} Z^{SU(2), N_{fl}=1}$ ``contains" the AD partition function

Extract it from the scaling region. Our result:

$$Z^{AD3} = \lim_{\Lambda_{AD} \to 0} \left(Z_u^{AD3-family} + Z_{SW}^{AD3-family} \right)$$

Claim: This is the AD3 TFT on X for $b_2^+ > 0$.

AD3 Partition Function: Evidence 1/2

Existence of limit is highly nontrivial. It follows from ``superconformal simple type sum rules" :

Theorem [MMP, 1998] If the superconformal simple type sum rules hold:

a.) $\chi_h - c^2 - 3 \leq 0$
b.) $\sum_\lambda SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} \lambda^k = 0 \quad 0 \leq k \leq \chi_h - c^2 - 4$

Then the limit $m \to m_*$ exists

It is now a rigorous theorem that SWST $\Rightarrow$ SCST

AD3 Partition Function: Evidence 2/2

For $b_2^+ > 0$ recover the expected selection rule:

$$\langle U(p)^{\ell} U(S)^{r} \rangle \neq 0 \text{ only for } \frac{6\ell + r}{5} = \frac{\chi_h - c^2}{5}$$

Consistent with background charge for AD3 computed by Shapere-Tachikawa.

Explicitly, if X of $SWST$ and $b_2^+ > 1$:

Explicitly, if X of $SWST$ and $b_2^+ > 1$:

$$Z^{AD3} = \frac{1}{\mathfrak{B}!} K_1 K_2^{\chi} K_3^{\sigma} \sum_{\lambda} e^{2\pi i\, \lambda\cdot w_2} SW(\lambda) \cdot$$

$$\cdot \{(\lambda\cdot S)^{\mathfrak{B}-2}(24(\lambda\cdot S)^2 + \mathfrak{B}(\mathfrak{B}-1)S^2\}$$

$$\mathfrak{B} = \chi_h - c^2 = -\frac{7\,\chi + 11\,\sigma}{4}$$

Surprise! p drops out: U(p) is a ``null vector''

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Z_u Wants To Be A Total Derivative

$$Z_u = \int \mathrm{d}u\, \mathrm{d}\,\bar{\mathrm{u}}\;\; \mathcal{H}(u)\;\; \Psi$$

$$\Psi = \sum_{\lambda} \frac{d}{d\bar{u}} \left(\mathcal{E}(r_\lambda^\omega, b_\lambda) \right) e^{-i\,\pi\tau\lambda^2 - i\,S\cdot\frac{du}{da} + 2\,\pi i\,(\lambda-\lambda_0)\cdot w_2}$$

$$\mathcal{E}(r,b) = \int_b^r e^{-2\pi t^2} dt \qquad r_\lambda^\omega = \sqrt{y}\lambda_+ - \frac{i}{4\pi\sqrt{y}} S_+ \frac{du}{da}$$

Lower limit b_λ must be independent of ū but can depend on λ, ω, u.

[Korpas & Manschot; Moore & Nidaiev]

$$Z_u \overset{?}{=} \int \mathrm{d}u\, \mathrm{d}\,\bar{\mathrm{u}} \frac{d}{d\bar{u}} \left(\mathcal{H}(u) \Theta^\omega \right)$$

No! $Z_u = \oint \mathrm{d}\,\bar{\mathrm{u}} \frac{d}{d\bar{u}} \left(\mathcal{H}(u) \Theta^{\omega} \right) = \oint d\bar{u}\, \mathcal{H} \Psi$

has nontrivial periods in general!

BUT: The difference for two metrics CAN be evaluated by residues!

$$Z_u^{\omega} - Z_u^{\omega_0} = \int du\, d\bar{u}\, \frac{d}{d\bar{u}} \left(\mathcal{H} \widehat{\Theta}^{\omega,\omega_0} \right)$$

$$\widehat{\Theta}^{\omega,\omega_0} = \sum_{\lambda} \left(\mathcal{E}(r_\lambda^{\omega}) - \mathcal{E}\left(r_\lambda^{\omega_0}\right) \right) e^{-i\,\pi\tau(u)\lambda^2 + \cdots}$$

$$r_\lambda^{\omega} = \sqrt{y}\lambda_+ - \frac{i}{4\pi\sqrt{y}} S_+ \frac{du}{da}$$

Modular completion of indefinite theta function of Vigneras, Zwegers, Zagier

Holomorphic Anomaly & Metric Dependence

For theories with a manifold of superconformal couplings, $\tau(u) \to \tau_0$ when $u \to \infty$

$$\widehat{\Theta}^{\omega,\omega_0}(\tau(u), \ldots) \to \widehat{\Theta}^{\omega,\omega_0}(\tau_0, \ldots)$$

Contour integral at $\infty \Rightarrow$ $For \quad b_2^+ = 1,$ many ``topological'' correlators:

<u>Nonholomorphic</u> in τ_0

Depend ***<u>continuously</u>*** on metric

[Moore & Witten, 1997 -- albeit *sotto voce*]

Special Case Of $\mathbb{CP}^2$

No walls: $\omega \in H^2(X,\mathbb{R}) \cong \mathbb{R}$, so we only see holomorphic anomaly.

$\Rightarrow$ ***Path integral*** derivation of the holomorphic anomaly.

Vafa-Witten Partition Functions

VW twist of N=4 SYM formally computes the ``Euler character'' of instanton moduli space.

(Not really a topological invariant. True mathematical meaning unclear, but see recent work of Tanaka & Thomas; Gholampour, Sheshmani, & Yau.)

Physics suggests the partition function is ***both*** modular (S-duality) ***and*** holomorphic.

Surprise! Computations of Klyachko and Yoshioka for $X = \mathbb{CP}^2$ show that the holomorphic generating function is only mock modular.

But a nonholomorphic modular completion exists.

This has never been properly derived from a path integral argument. .

But we just derived completely analogous results for the $N_{fl} = 4$ and $\mathcal{N} = 2^*$ theories from a path integral, suggesting VW will also have continuous metric dependence.

Indeed, continuous metric dependence in VW theory for $SU(r > 1)$ has been predicted in papers of Jan Manschot using rather different methods.

We hope that a similar path integral derivation can be found for the $\mathcal{N} = 4$ Vafa-Witten twisted theory.

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Future Directions

Lots of details to clean up

Comparison with localization results of Gukov et. al.

There is an interesting generalization to invariants for families of four-manifolds.

Generalization to all theories in class S:
Many aspects are clear – this is under study.

Does the u-plane integral make sense for ***ANY*** family of Seiberg-Witten curves ?