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The Impact of D-branes on Mathematics

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Abstract: This records a few thoughts I had preparing for a panel discussion on the occasion of the JOEFEST conference at the KITP. March 2, 2014

Contents

1. Introduction

I was asked to comment on the role D-branes have played in developments in Mathematics. Here I respond with a few highly biased impressions.

Let me begin with three preliminary remarks. First, I come to view D-branes through the lens of Mathematics, not to praise them. The good that men do lives after them, and others at this conference will bestow the praise D-branes so richly deserve. I suspect that the expectation value of the influence of D-branes over the entire ensemble of mathematicians is extremely small, but if we restrict the ensemble to mathematicians who work on geometry, topology and representatation theory, and then weight the average by the quality of the mathematician we'll get a very significant signal. Second, I will try to hew closely to the specific influence of D-branes *per se*, rather than talking about the broader subject of physical mathematics. That is somewhat arbitrary and unnatural because of course D-branes are so closely interrelated with other branes and solitons, and physical mathematics more broadly. Third, we should distinguish two different phenomena:

1. Physicists have used the D-brane construction to understand better previously existing Mathematics.

2. Mathematicians and physicists have used the D-brane construction to discover new results and insights in Mathematics.

Again this dichotomy is somewhat unnatural since, once physicists have understood some old but significant mathematics, their very new perspective often leads to very new and unexpected mathematics. It is a good example of what I have called the unreasonable effectiveness of physics in mathematics [18].

A good example of the above distinction is provided by the ADHM construction of instantons and Nahm's construction of monopoles. For a physicist these become much more transparent by thinking about, say, the D0D4-brane or D1D3 systems, respectively. Putting branes in orbifolds then leads very naturally to the Kronheimer-Nakajima construction of instantons on ALE spaces. But then, the physicists come back with new results: Branes on orbifolds led to lots of work on quivers, which in turn has had a deep influence on certain aspects of representation theory, specifically on the study of varieties of quiver representations. Another example might be Cherkis' bow construction of instantons on ALF spaces, and yet another is Tachikawa's recent progress on SO(N) instantons on ALE spaces, etc. More broadly speaking, physicists have generalized and geometrized many constructions involving instantons and monopoles in novel ways.

But this is just one case in point. Attempting to take a bird's eye view I can discern five broad influences of D-branes on Mathematics. To reiterate: The distinctions are rather artificial since the five influences are all closely related.

1. Duality symmetries, D-brane categories, and extended field theory
2. Noncommutative geometry
3. Twisted K-theory, and differential cohomology
4. Analytic number theory (via asymptotics of Fourier coefficients of automorphic forms, Rademacher summability, mock modular forms)
5. F-theory and the algebraic geometry of elliptic fibrations

2. Duality symmetries, D-brane categories, and extended field theory

The duality symmetries here are primarily mirror symmetry and S-duality. D-branes are connected to two central developments in this area of Mathematics:

1. Homological mirror symmetry and stability conditions on D-brane categories
2. Geometric Langlands Program via S-duality of N=4 SYM and geometric representation theory,

Maybe it is worth elaborating on these a little bit:

The homological symmetry conjecture was first articulated by Maxim Kontsevich at the 1994 International Congress of Mathematicians in 1994. Now, Joe's paper [22] pointing out that D-branes carry RR charge was from October 1995. Thus, the claim that D-branes influenced homological mirror symmetry might seem to imply closed timelike loops. It does not. Homological mirror symmetry was not well understood by many people when it was announced. Certainly not by any physicists. Indeed it took quite a while for physicists to understand what Kontsevich had said, and it was essential to have D-branes and the Strominger-Yau-Zaslow picture of mirror symmetry [26] (which was again rather directly motivated by T-duality properties D-branes) before physicists could begin to understand. An anecdote will illustrate this I wrote a paper with Jeff Harvey in September 1996 discussing the relation of D-branes to coherent sheaves [11] and Kontsevich wrote to me saying, in effect: "You dope: You should be using the *derived category* of coherent sheaves." I believe it was only after Douglas pointed out that the complexes are graded by $U(1)_R$ charge of the N=2 worldsheet theory, and that these charges have physical implications for open string tachyon masses, that physicists really started to take the derived category more seriously. However, once the physicists did understand that there are categories of A-branes and B-branes which should be equivalent there was a flowering of work and progress, even in the purely mathematical questions about these categories. To choose but one example, Hori-Iqbal-Vafa's work [13] on D-branes in 2d LG models led to some important progress in understanding the Fukaya-Seidel category of A-branes on Lefshetz fibrations.

D-branes and the SYZ picture of mirror symmetry really opened up the way for the Kapustin-Witten interpretation of the Geometric Langlands Program in terms of the S-duality of N=4 SYM [14]. We had known for a long time that S-duality was related to mirror symmetry of a 2d sigma model whose target space is the moduli of Hitchin systems [2, 12]. But when Kapustin and Witten put D-branes on these moduli spaces the picture really came together nicely. The relation of D-branes to noncommutativity was also important in the Kapustin-Witten story (through the relation to D-modules), and I'll come back to this issue.

In any case, the Kapustin-Witten approach to the Geometric Langlands Program has had a tremendous and powerful influence on what mathematicians call "Geometric Representation Theory." This is a subject where geometry illuminates representation theory and vice versa. A paradigmatic example of a result in this field is the Borel-Weil-Bott theorem. The first example of that theorem is the statement that the Landau levels of an electron confined to a sphere surrounding a charge m monopole transform in an irreducible representation of $SU(2)$ of dimension $|m|+1$. One leader in the field, David Ben-Zvi, told me that *"MY perspective on geometric rep theory is completely colored by D-branes."* Since geometric representation theory is such an active subject right now with lots of papers, conferences, MSRI programs etc., we can see that just this one line of development, D-branes $\rightarrow$ Geometric Langlands Program via S-duality, has had a deep influence on Mathematics.

Another thing that came out of all the feverish work around D-branes in the late 1990's was the notion of *stability conditions.* Through a happy coincidence mathematicians' use of a term "stability of vector bundles" was not too far removed from physicists' understanding of stability of physical states against decay into constituents. Stability conditions were implicit in much work at the end of the 1990's and played an important role in Douglas' work on derived categories and D-branes. They also played an absolutely critical role in my work with Frederik Denef trying to prove the OSV conjecture [7]. From the mathematical viewpoint it became crucial to Kontsevich-Soibelman's work on trying to refine the SYZ picture of mirror symmetry into a statement one could really prove. But then in turn the whole subject of wall-crossing has taken on a vigorous life of its own, motivated both from physical and mathematical considerations. This is perhaps the liveliest area of interaction between physics and mathematics at the time of writing. So here again, D-branes played a crucial link in the evolutionary history of ideas in physical mathematics, and Mathematics more generally.

3. Noncommutative geometry

Again, we must come back to the distinction mentioned at the beginning. In groundbreaking work by Nekrasov-Schwarz [21], Connes-Douglas-Schwarz [4], and Seiberg-Witten [24] old constructions of noncommutative geometry were given new life through D-branes. [1] The work of CDS and SW in particular made very explicit use of the D-brane construction. Physicists really understood very well the Moyal plane and the noncommutative torus, in particular the work of Connes and Rieffel on these topics.

But then they really gave something in return: As I have learned most cogently from Graeme Segal, one of the great lessons, for Mathematics, of string theory more broadly is that we should replace spacetime by some more algebraic notion of a string background and that this string background is, in some sense, noncommutative. Our usual conception of spacetime is a derived concept. See, for example, [23]. We replace the notion of spacetime by a category of branes, and then the closed string sector and ultimately the notion of a pseudoriemannian manifold of spacetime should be derived from that. [2] As Segal points out, there is a nice analog of this in homotopy theory: From a category you can produce a space. If you have a manifold with open cover there is a naturally associated category and the space of that category has the same homotopy type as the space you started with. String theory is providing some generalization of that.

So, perhaps the most central influence of D-branes on Mathematics is this: [3] A natural development following from the D-brane construction is that to a string background - as we currently understand it - one should associate a *category* of D-branes. Such categories arise in many different ways, for example through representations of quivers, derived categories of sheaves on algebraic varieties, Fukaya categories, and so on. These categories are regarded as interesting by, say, number theorists who work in class-field theory or by symplectic geometers and pure algebraic geometers who have no interest in physics.

The influence of D-branes on noncommutative geometry and algebraic geometry sometimes leads to strange juxtapositions of names. For example in a series of papers Yau and Liu refer to the "Grothendieck-Polchinski ansatz." A stranger combination of bedfellows is hard to imagine. [4]

[1] My impression is that the Nekrasov-Schwarz work was not as much motivated by D-branes, but it played a critical role in this development.

[2] The fact that D-branes should be objects in a category follows from very simple and basic sewing arguments [20]. In the context of open/closed 2d TFT one sees very beautifully and simply how the closed string sector is derived from the open string sector through cyclic cohomology of the category of branes.

[3] Here I am paraphrasing things Graeme said to me in a recent exchange on the topic of this essay.

[4] Incidentally, it should probably be called the "Grothendieck-Polchinski-Witten" ansatz, since it was

4. Twisted K-theory, and differential cohomology

D-branes are of course sources of RR fields. This was one of the deep and central points of Polchinski's paper [22]. As a consequence, there has also been some mathematical influence through understanding better the mathematical nature of RR fields.

This leads to the topic of K-theory, particularly twisted K-theory. Mathematicians knew about twisted K-theory long ago but the subject got a tremendous boost from D-branes and has flowered and developed well since. One example of the recent advances is a beautiful result of Freed-Hopkins-Teleman identifying certain twisted K-theories of Lie groups with the Verlinde algebra [9]. [5]

Careful examination of the RR sector in the light of D-branes also had a role in the development of differential K-theory, a generalization of Deligne-Cheeger-Simons cohomology. Hopkins and Singer developed a general theory of differential (generalized) cohomology. It was mostly motivated by Witten's paper on the M5-brane partition function [28], but it turns out that, mathematically speaking, RR currents, RR charges, and RR fields find their natural mathematical home in differential K-theory.

When people started incorporating orientifolds into the story it seemed that physicists were inventing lots of new flavors and generalizations of K-theory and thereby innovating substantially on that venerable subject. But, as I learned in work with Jacques Distler and Dan Freed, really the zoo of K-theories appropriate to classifying D-brane charges in orientifolds are all just different versions of twisted equviariant K-theory. Nevertheless, things like T-duality and M/type IIA duality have very striking implications for the subject of K-theory.

5. Analytic Number Theory

Natural enumerative problems are associated to categories of Branes. A good example are the Donaldson-Thomas invariants which for mathematicians are certain enumerative invariants associated to the derived category of sheaves and for physicists have to do with counting BPS states in supersymmetric field theories or in supersymmetric compactifications of type II string theories on Calabi-Yau manifolds. Often, interesting analytic functions are made from generating functions of these enumerative counts, and they often turn out to have automorphic properties thanks to duality symmetries. (The paradigmatic example of black hole entropy counting - started by Strominger and Vafa [25] - is a direct outgrowth of Polchinski's paper.) But these so-called automorphic functions or forms actually push the boundary of what we understand in that subject. In some circumstances there is a conflict between holomorphy and duality. This was noticed in [16] and further

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Witten's paper [27], following closely on the heels of Polchinski's, which made the striking point that with stacks of D-branes the normal bundle scalar fields become matrices of higher rank, i.e. "noncommutative coordinates," thus geometrizing adjoint Higgs breaking of $U(N)$ to its maximal cartan subgroup and providing the first crucial step on the way to noncommutative geometry.

[5] For physicists we are talking about symmetry-preserving branes in WZW models. These wrap special conjugacy classes associated with the integrable highest weight representations of loop groups. See [19] for the physical perspective.

developed by Manschot [17]. It was also noted by Gaiotto, who gave a talk on the subject, but as far as I know never published anything. The path leads directly to mock modular forms which appeared, in all but name, in [16]. The applications of mock modular forms to four-dimensional black holes in N=4 supergravity was explored extensively and in great depth by Dabholkar-Murthy-Zagier [5]. In this way, new results on "mock modular forms" and asymptotics of fourier coefficients of certain related automorphic forms are being motivated, ultimately, by the D-brane construction.

These new developments in applications of analytic number theory have been beautifully combined with the discovery of Mathieu Moonshine [8], and generalized to "Umbral Moonshine" by Cheng-Duncan-Harvey [3]. These are certainly striking new results in the theory of automorphic forms and sporadic groups. And some very concrete beautiful new results are coming out of that work. To choose just one example: Cheng-Duncan-Harvey give a quite new construction of the Niemeier lattices!

6. F-Theory

I will here be brief because of lack of knowledge. The subject is immense. Tremendous work in algebraic geometry has been motivated by F-theory, again made possible by the peculiar properties of (p, q) 7-branes. (The D7-brane is the case $p = 1, q = 0$.) For examples, there have been beautiful new mathematical results on the classification of commuting triples in Lie algebras [6] and the construction of holomorphic vector bundles on elliptic fibrations [10].

7. D-branes and mathematics of the future

Of course, the most interesting aspect of the question of this essay is what D-branes will contribute to Mathematics in the future.

There is a plethora of tasty, bite sized projects that you might consume yourself or share with a few friends. To choose but one from a long list of examples, I hope that D-branes will shed some light on the still-mysterious Mathieu Moonshine and Umbral Moonshine phenomena. The Mathieu group should be a quantum symmetry of the BPS sector which should only become apparent when studying the *target space* D-branes. Roughly speaking, M_{24} should be the automorphism group of some kind of "algebra of BPS states." No one has, as yet, managed to make this intuition precise.

Then there are the broader philosophical questions - more like the giant birthday cake sized questions that only a full community could possibly consume:

1. Mathematicians have understood free field theory and (extended) topological field theory very well, but they don't understand scale, the renormalization group, and interacting field theories very well. But mathematicians are very good at geometry, and D-branes geometrize many aspects of quantum field theory. For example, Witten's derivation of the N=2 d=4 QFT's from M-theory (deriving the set of N=2 d=4 theories now called "class S") used a geometrization of the renormalization group in

terms of "brane bending" in an essential way [29]. Perhaps such a geometrization of field theoretic phenomena will lead to new and fruitful mathematical approaches to issues of scale and interaction in field theory. Here is a point where D-branes could yet contribute substantially to new and important Mathematics.

2. Returning to the central point we discussed in the section on noncommutative geometry, spacetime should be replaced by D-brane categories. But these categories are more naturally associated to spaces which are noncommutative. Exploring the relation between emergent spacetime and D-brane categories is likely to remain a basic mainstay of progress for the forseeable future.

3. And then there is the big elephant in the room: We still have no fundamental formulation of string theory. Perhaps the best attempt to date is the Matrix theory [1]. This approach makes essential use of D-branes. But it is only an attempt, and it has some flaws. (Among them, it has not led to much significant new mathematics.) Searching for a satisfactory formulation of string/M-theory is surely going to lead to new and novel mathematics. Regrettably, it is a problem the community seems to have put aside - temporarily. But we should not lose sight of it.

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