Testing Some
Black Hole/Topological String
Conjectures

Gregory Moore

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Introduction

We will be exploring some relations between black hole entropy and topological string theory.

The setting is type II string theory compactified on

$$\mathbb{R}^{1,3} \times \mathcal{X}$$

with $\mathcal{X}$ a smooth compact Calabi-Yau manifold.

The work of

MSW: Maldacena, Strominger, Witten

CDM: Cardoso, DeWit, Mohaupt

OSV: Ooguri, Strominger, Vafa

in this setting led to some striking conjectures for relations between entropy of BPS black holes and topological string theory on $\mathcal{X}$.

We will be trying to test some of these conjectures.

Summary

We focus on BPS states with charges denoted (p, q).

For large charges, these states are semiclassically black holes.

The attractor mechanism expresses the entropy $S(p, q)$ of the black hole in terms of a prepotential entering the 4d effective supergravity.

This led OSV to propose a relation of the form

$$\boxed{\Omega(p,q) = \int d\phi \; |\Psi_{top}(p+i\phi)|^2 e^{\pi q\cdot\phi}}$$

1. We will make the proposal more precise, and try to define our terms carefully.

2. We will show that one can systematically evaluate the integral in some limits.

3. We can then test the proposal by using typeII/heterotic duality to compare to known degeneracies of perturbative heterotic BPS states.

4. We find that sometimes the proposal works, (in a highly nontrivial way!) and sometimes it appears to fail.

$\mathcal{N} = 2$ supergravity

Low energy type II compactified on a smooth $\mathcal{X}$ is $\mathcal{N} = 2$ supergravity coupled to vectormultiplets in an abelian gauge group.

The group of charges is

$K^1(\mathcal{X})$ for IIB $\qquad\qquad$ $K^0(\mathcal{X})$ for IIA.

For today's talk we focus on IIA and replace $K^0(\mathcal{X})$ by

$H^{\mathrm{even}}(\mathcal{X}) = [H^0(\mathcal{X}) \oplus H^2(\mathcal{X})]_{\mathrm{magnetic}} \oplus [H^4(\mathcal{X}) \oplus H^6(\mathcal{X})]_{\mathrm{electric}}$

The effective $\mathcal{N} = 2, d = 4$ supergravity has h vectormultiplets,

$h = h^{1,1}(\mathcal{X})$ for IIA and $h = h^{2,1}(\mathcal{X})$ for IIB

with moduli described by special coordinates X^I, $I = 0, 1, \ldots, h$

For charges (p, q) one finds BPS black hole solutions.

The values of $g_{\mu\nu}$ and X^I at the horizon are functions of (p, q) ("Attractor mechanism")

Entropy & Microscopics

From the supergravity solution one computes the entropy via Beckenstein-Hawking:

$$\boxed{S = \frac{A}{4\pi} = |Z(p,q)|^2}$$

How do we account for this microscopically?

MSW showed (with no D6 branes) that one can do so by interpreting the configuration as a wrapped M5 brane.

One finds

$$\underline{S = |Z(p,q)|^2 + \text{correction}}$$

The correction is due to $\int R^2$ in $\mathcal{L}_{eff}$.

A formula for the entropy

CDM investigated corrections to the area law from higher derivative terms in the effective action of the supergravity, focussing on those from chiral integrals over superspace.

The chiral superspace terms are governed by a prepotential

$$F_{\text{sugra}}(X^I, W^2)$$

Weyl multiplet

$$W^2 = (T^-)^2 + \cdots + \theta^4 ({}^-C^-)^2 + \cdots$$

The prepotential has an expansion

$$F_{\text{sugra}} = \sum_{h \geq 0} F_h(X^I) W^{2h}$$

So the higher-derivative terms in the effective action form an expansion

$$\mathcal{L}_{eff} \sim \sum_{h \geq 1} \int ({}^-C^-)^2 (T^-)^{2h-2}$$

Reformulation in terms of a free energy

The formula of CDM was expressed by OSV as follows:

Define:

$$\boxed{\mathcal{F}(\phi, p) := -\pi \mathrm{Im}\Big[F_{\mathrm{sugra}}(p^I + i\phi^I, 2^8)\Big]}$$

ϕ^I is real, = electric potential, = RR potential.

Then the "chiral Wald entropy" S_{bh} is:

$$S_{bh}(p,q) = \mathcal{F}(\phi,p) - \phi^I \frac{\partial \mathcal{F}}{\partial \phi^I}$$

$$\frac{\partial \mathcal{F}}{\partial \phi^I} = -\pi q_I$$

The OSV proposal

Based on the formula

$$S_{bh}(p,q) = \mathcal{F}(\phi,p) - \phi^I \frac{\partial \mathcal{F}}{\partial \phi^I}$$

OSV went further and proposed that:

1. One should introduce a mixed thermodynamical ensemble

$$Z_{bh}(p,\phi) := \sum_q \Omega(p,q) e^{-\pi q \phi}$$

2. Since

$$\Psi_{top} \sim e^{iF_{\rm sugra}}$$

they then conjectured:

$$|\Psi_{top}|^2 = Z_{bh}$$

This is formally equivalent to

$$\Omega(p,q) = \int d\phi \; |\Psi_{top}(p+i\phi)|^2 e^{\pi q \cdot \phi}$$

This is the conjecture we wish to test.

A few problems

1. The sum Z_{bh} does not converge. The physical ensemble is unstable.

2. Formally, Z_{bh} is periodic in ϕ, but $\Psi_{top}(p + i\phi)$ is not.

These two problems are ameliorated by the integral form of the conjecture, so we focus on that.

3. What is Ψ_{top} ?

4. What is the integration domain (which contour)?

5. What is Ω? How do we count the BPS states?

There are nontrivial issues associated with each of these.

What is Ψ_{top}?

Except in rare cases $F_{\rm sugra}$ is only known as an asymptotic expansion near large radius limits $\Rightarrow$ Distinguished X^0:

$$X^I = (X^0, X^A), \qquad A = 1, \dots, h$$

$$\boxed{F_{\rm sugra} = F^{\rm pert} - \frac{2i}{\pi} F^{GW}(\lambda, q)}$$

$$F^{\rm pert} = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} - \frac{c_{2A} X^A}{6 X^0}$$

$$F^{GW}(\lambda, q) = \sum_{h \geq 0, \beta} N_{h,\beta}\, q^\beta\, \lambda^{2h-2}$$

$$\beta \in H_2(\mathcal{X}), \qquad q^\beta = e^{2\pi i \int_\beta (B + iJ)} \sim e^{-\text{Area}(\beta)}$$

$$\lambda = \frac{4\pi}{X^0}$$

$$\boxed{\Psi_{\rm top}(X) := \lambda^{\chi/24} e^{-\frac{i\pi}{2} F_{\rm sugra}}}$$

1. $F^{GW}(\lambda, q)$ is only defined as an asymptotic series for $\lambda \to 0$. We assume there is some well-defined nonperturbative completion.

2. The prefactor $\lambda^{\chi/24}$ is inserted by hand to make things work better. It is similar to a factor needed for the nonholomorphic version of $\Psi_{\rm top}$ to satisfy the BCOV equation.

Evaluation in the perturbative approximation - I

For appropriate charges there is a systematic saddle point evaluation of the integral keeping only $F^{\rm pert}$.

For simplicity, set $p^0 = 0 \Rightarrow$

$$t^A = \frac{X^A}{X^0} = \frac{p^A + i\phi^A}{i\phi^0} = \frac{\phi^A}{\phi^0} - i\frac{p^A}{\phi^0}$$

Keep perturbative part

$$\boxed{\mathcal{F}^{\rm pert} = -\frac{\pi}{6}\frac{\hat{C}(p)}{\phi^0} + \frac{\pi}{2}\frac{1}{\phi^0}C_{AB}\phi^A\phi^B}$$

$C(p) := C_{ABC}p^A p^B p^C, \quad \hat{C}(p) := C(p) + c_{2A}p^A, \quad C_{AB} := C_{ABC}p^C$

Saddle point analysis for

$$\int d\phi e^{\mathcal{F}^{\rm pert} + \pi q\phi}$$

$$\phi^0_* = -\sqrt{\frac{\hat{C}(p)}{6|\hat{q}_0|}}, \qquad {\rm Im}t^A_* = -\frac{p^A}{\phi^0_*}$$

$$\hat{q}_0 = q_0 - \frac{1}{2}q_A C^{AB} q_B$$

Evaluation in the perturbative approximation - II

Gaussian integral on $\phi^A \Rightarrow$

$$\int d\phi e^{\mathcal{F}^{\rm pert}+\pi q\phi} \to \mathcal{N}(p)\int d\phi^0 (\phi^0)^h e^{-\frac{\hat{C}}{\phi^0}+\hat{q}_0\phi^0}$$

$\Rightarrow$ Bessel integral $\Rightarrow$ asymptotics

$$\longrightarrow \quad \mathcal{N}(p)\hat{I}_\nu\left(2\pi\sqrt{\frac{\hat{C}(p)|\hat{q}_0|}{6}}\right) \qquad \nu = h/2+1$$

Roughly

$$\underline{\hat{I}_\nu(z) \sim e^z}$$

More precisely:

$$\hat{I}_\nu(z) \sim e^z \cdot z^{-\nu-1/2}\big[1+\mathcal{O}(1/z)\big]$$

$$\sim e^z \cdot z^{-\nu-1/2}\left[1-\frac{(4\nu^2-1)}{8z}+\frac{(4\nu^2-1)(4\nu^2-3^2)}{2!(8z)^2}+\dots\right]$$

Justifying the saddle-point evaluation

$$\boxed{\begin{array}{c} \phi^0_* = -\sqrt{\dfrac{\hat{C}(p)}{6|\hat{q}_0|}}, \qquad \mathrm{Im} t^A_* = -\dfrac{p^A}{\phi^0_*} \\ \\ |\hat{q}_0| \gg \hat{C}(p) \qquad \Rightarrow \qquad |\phi^0_*| \ll 1 \end{array}}$$

So if $\beta \neq 0$ we have

$$|(q^\beta)_{s.p.}| \ll 1$$

so terms in F^{GW} with $\beta \neq 0$ contribute exponentially small corrections.

However we must handle the $\beta = 0$ contributions to F_{GW} separately.

We will show that in fact these contributions are also exponentially small.

N.B.! $\lambda \sim 1/\phi^0$ is therefore large - we are in the nonperturbative regime of the topological string!

Handling the $\beta = 0$ terms

GV defined a nontrivial rearrangement of the GW series:

$$\sum_{h\geq 0,\beta} N_{h,\beta}\, q^\beta\, \lambda^{2h-2} = \sum_{h\geq 0,\beta,d\geq 1} n_\beta^h \frac{1}{d}\left(2\sin\frac{d\lambda}{2}\right)^{2h-2} q^{d\beta}$$

$\Rightarrow$ For $\beta = 0$ the GW series can be summed:

$$\boxed{\begin{gathered}\sum_{h\geq 0} N_{h,0}\lambda^{2h-2} \sim n_0^0 f(\lambda)\\[1em] f(\lambda) := \sum_{d=1}^{\infty} \frac{1}{d}(2\sin\frac{d\lambda}{2})^{-2} = \log\prod_{k=1}^{\infty}(1-e^{i\lambda k})^k\end{gathered}}$$

and $n_0^0 = -\frac{1}{2}\chi$.

$\lambda = \frac{4\pi}{i\phi^0}$, $\phi^0 < 0$ small $\Rightarrow$ $f(\lambda)$ is exponentially small.

Actually, we need a slight correction:

$$n_0^0\left[f(\lambda) + \frac{1}{12}\log\frac{\lambda}{2\pi i} - K\right] \sim \sum_h N_{h,0}\lambda^{2h-2}$$

$\Rightarrow$ correction factor $\lambda^{\chi/24}$.

Summary

We do not know Ψ_{top} nonperturbatively.

We assume some nonperturbative completion exists.

We can write

$$\mathcal{F} = \mathcal{F}^{\text{pert}} + \mathcal{F}^{GW}_{\beta=0} + \mathcal{F}^{GW}_{\beta\neq 0}$$

We *assume* that $\mathcal{F}^{GW}_{\beta\neq 0} \sim e^{-A}$ if worldsheet instantons all have area $\geq A$, even at strong string coupling $\lambda \sim A$. Note: $\lambda^{2h-2}q^{\beta} \sim A^{2h-2}e^{-A}$.

In this case the asymptotics of

$$\int d\phi \, |\Psi_{top}(p+i\phi)|^2 e^{\pi q\cdot\phi}$$

for

$$|\hat{q}_0| \gg \hat{C}(p)$$

is that of

$$\boxed{\mathcal{N}(p)\hat{I}_\nu\left(2\pi\sqrt{\frac{-\hat{C}(p)\hat{q}_0}{6}}\right) + \mathcal{O}\left(e^{-\sqrt{|\hat{q}_0|}}\right)}$$

Now, we want to test this using some known degeneracies.

Comparison with attractor formula

Compare Attractor formula

$$S(q,p) = 2\pi\sqrt{\frac{\hat{C}(p)|\hat{q}_0|}{6}} + \mathcal{O}(e^{-\kappa\sqrt{|\hat{q}_0|}})$$

Remark: DeWit et. al. and Sen take a different approach. They are trying to produce the microcanonical degeneracies

$$S_{\mathrm{micro}}(p,q) := \log\Omega(p,q)$$

from

$$S(p,q) = \mathcal{F}_{eff}(\phi,p) - \phi^I \frac{\partial \mathcal{F}_{eff}}{\partial \phi^I}$$

$$\frac{\partial \mathcal{F}_{eff}}{\partial \phi^I} = -\pi q_I$$

What is Ω?

Various candidates

$\mathcal{H}_{BPS}(Q) :=$ Hilbert space of BPS states of charge Q.

$$\Omega_{abs}(Q) := \dim \mathcal{H}_{BPS}$$

Helicity Supertraces:

$$\Omega_n(Q) := \mathrm{Tr}_{\mathcal{H}_{BPS}(Q)}(-1)^{2J_3}(J_3)^n$$

J_3 in the massive little group in 4 dimensions.

Moduli dependence

1. $\Omega_{abs}(Q)$ is only *locally* constant as a function of moduli

2. $\Omega_2(Q)$ is constant as a function of hypermultiplet moduli, but can jump across lines of marginal stability

3. $\Omega(Q)$ can depend on (K-theory) torsion charges

Specialization: Small black holes in $K3$-fibrations

Now, specialize to cases where $\Omega(p,q)$ is exactly computable from a dual picture.

$\mathcal{X}$ is K3-fibered.

$D4$ wraps the $K3$ fibers. $D2$ wraps cycles in $K3$.

For such charges one has $C(p) = C_{ABC}p^A p^B p^C = 0$, but

$$\hat{C}(p) = c_{2A}P^A = 24p^1 \neq 0$$

The advantage of these states is that under Heterotic/TypeII duality they map to perturbative heterotic BPS states.

These states are known as "small black holes": The area of the horizon comes from the 1-loop term in the prepotential. In terms of supergravity the R^2 corrections to the action are important. [Dabholkar; Dabholkar, Kallosh, Maloney; Sen].

Whether or not there is a consistent interpretation in terms of black holes, we may regard the OSV proposal as a statement about BPS degeneracies, and test it.

Example: Het/T^6= TypeII/$K3 \times T^2$

The small black holes map to Dabholkar-Harvey states.

These are BPS states with momentum $-q_0$ and winding p^1 around a circle in T^6.

These have the form

$$\prod \alpha_{-n}^{N_n}|0\rangle_{\rm left} \otimes |Q_L; Q_R\rangle \otimes |8_v + 8_s\rangle_{\rm right}$$

with

$$N - 1 = \frac{1}{2}(Q_R^2 - Q_L^2) = \frac{1}{2}Q^2 = -p^1 q_0$$

$\Rightarrow$

$$\boxed{\begin{gathered} \Omega_{abs}(Q) = p_{24}(\frac{1}{2}Q^2) \\ \frac{1}{\eta^{24}} = \frac{1}{q\prod(1-q^n)^{24}} = \sum_{N=0}^{\infty} p_{24}(N) q^{N-1} \end{gathered}}$$

Comparing OSV with exact degeneracy

Compare type II

$$2\pi\sqrt{\hat{C}\hat{q}_0/6} = 4\pi\sqrt{N-1}$$

$$\boxed{\begin{array}{rl} \text{OSV}: & \Omega(p^1, q_0) \sim \mathcal{N}(p)\hat{I}_{13}(4\pi\sqrt{N-1}) \\ \text{Exact}: & \Omega_{abs}(Q) = p_{24}(N) \end{array}}$$

There is an *exact* formula (Rademacher formula):

$$p_{24}(N) = 16\cdot\left[\hat{I}_{13}(4\pi\sqrt{N-1}) + 2^{-14}e^{i\pi N}\hat{I}_{13}(2\pi\sqrt{N-1}) + \cdots\right]$$

Spectacular agreement of all orders in $1/N$ in the leading exponential.

Do the subleading terms agree?

In fact, Ψ_{top} is known exactly for $K3 \times T^2$:

$$|\Psi_{\rm top}|^2 = e^{\frac{\pi}{2}p^1 C_{AB}\phi^A\phi^B/\phi^0}|\Delta(\tau)|^{-2}$$

$$\tau = (\phi^1 - ip^1)/\phi^0$$

$\Rightarrow |\Psi_{\rm top}|^2$ is not a normalizable wavefunction

Generalizations

We can carry out this strategy for more general Het/TypeII pairs.

We focus on orbifolds of Het/T^6.

These are limits of Het/$K3 \times T^2$ and we have DH states with momentum + winding around T^2.

Such BPS states are in:

$$\mathcal{H}_{osc,L} \otimes \mathcal{H}_{mom} \otimes \tilde{\mathcal{H}}_{gnd}$$

We can compute $\Omega_{abs}(Q)$ and $\Omega_n(Q)$ in terms of the coefficients of modular forms for congruence subgroups of $SL(2, \mathbb{Z})$.

We then find the asymptotics of $\Omega(Q)$ using the "Rademacher expansion."

Comparing with the OSV prediction we find partial agreement:

1. Reduced rank $\mathcal{N} = 4$ models: Good agreement

2. $\mathcal{N} = 2$ heterotic orbifolds: Good agreement in twisted sectors, discrepancies in untwisted sectors.

Degeneracies of DH states

There is an exact formula for the degeneracies of DH states in T^6/Γ orbifolds of charge Q:

$$\Omega_w(Q) = e^{4\pi Q_R^2} \int d\tau_1 \; q^{\frac{1}{2}Q_L^2} \bar{q}^{\frac{1}{2}Q_R^2} \mathcal{Z}_w$$

$$\mathcal{Z}_w = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \frac{1}{\eta^{2+2k}} \left[\prod_{j=1}^{11-k} (-2\sin\pi\theta_j(g)) \frac{\eta}{\vartheta[{\frac{1}{2} \atop \frac{1}{2}+\theta_j(g)}](|\tau)} \right] w(g) \mathcal{F}_{g,Q}(q)$$

$$w(g) = \begin{cases} 16\cos\pi\tilde{\theta}_1(g)\cos\pi\tilde{\theta}_2(g)\cos\pi\tilde{\theta}_3(g) & w = abs \\ 2(\sin\pi\tilde{\theta}(g))^2 & w = 2 \\ \frac{3}{2} & w = 4 \end{cases}$$

The Rademacher expansion- I

Consider a modular form

$$\boxed{F(\tau) = q^{\Delta} \sum_{n \geq 0} F(n) q^n}$$

with $\Delta < 0$ and $w < 0$:

$$F(-1/\tau) = (-i\tau)^w F(\tau)$$

Then we get the asymptotics of $F(n)$ for large n by:

$$F(n) = \int_{\tau_0}^{\tau_0+1} d\tau q^{-\Delta-n} F(\tau)$$

$$= \int_{\tau_0}^{\tau_0+1} d\tau e^{-2\pi i\tau(\Delta+n)} e^{-\frac{2\pi i\Delta}{\tau}} (-i\tau)^{-w}(1+\cdots)$$

$\Rightarrow$ saddle point $\tau = i\sqrt{\frac{|\Delta|}{n+\Delta}} \Rightarrow$

$$\boxed{F(n) \sim \hat{I}_{1-w}(4\pi\sqrt{|\Delta|(n+\Delta)})}$$

More generally,

$$F_i(-1/\tau) = (-i\tau)^w S_{ij} F_j(\tau)$$

$$F_i(n) \sim \sum_{\Delta_j < 0} S_{ij} I_{1-w}(4\pi\sqrt{|\Delta_j|(n+\Delta_i)})$$

The Rademacher expansion- II

In fact there is an exact expression:

$$F(n) = \sum_{s=1}^{\infty} K_{s,n} \hat{I}_{1-w}\left(\frac{4\pi}{s}\sqrt{|\Delta|(n+\Delta)}\right)$$

$$K_{s,n} \sim s^{w-3/2}$$

Example: FHSV model $(K3 \times T^2)/\mathbb{Z}_2$

Electric charges in untwisted sector

$$M_0 = \frac{1}{\sqrt{2}} II^{9,1} \oplus II^{1,1}$$

Electric charges in twisted sector:

$$M_1 = \frac{1}{\sqrt{2}} II^{9,1} \oplus (II^{1,1} + \delta)$$

$$\Omega_{abs}(Q) \sim \begin{cases} \hat{I}_{\nu}(4\pi\sqrt{\frac{1}{2}Q^2}) & Q \in M_0' \\ 0 & Q \in M_0 - M_0' \\ \hat{I}_7(4\pi\sqrt{\frac{1}{2}Q^2}) & Q \in M_1 \end{cases}$$

$\nu = 13$, for generic moduli, but can vary

$$\Omega_2(Q) \sim \begin{cases} e^{2\pi i Q\cdot\delta}(1 - e^{i\pi Q^2/2})\hat{I}_7(2\pi\sqrt{\frac{1}{2}Q^2}) & Q \in M_0' \\ 0 & Q \in M_0 - M_0' \\ \hat{I}_7(4\pi\sqrt{\frac{1}{2}Q^2}) & Q \in M_1 \end{cases}$$

Compare OSV prediction:

$$\Omega(Q) \sim \hat{I}_7(4\pi\sqrt{\frac{1}{2}Q^2})$$

for all Q.

Other Apparent Discrepancies

1. In general, for T^6/Γ orbifolds, for states in the twisted sector $\Omega_{abs} \sim \Omega_2$ goes like

$$\hat{I}_{\frac{1}{2}(n_v+2)}(4\pi\sqrt{\frac{1}{2}Q^2})$$

in nice agreement with OSV

However, for states in the untwisted sector $\Omega_{abs}(Q)$ is a function of hypermultiplet moduli,

while

$$\Omega_2(Q) \sim \hat{I}_{\frac{1}{2}(n_v+2)}(4\pi\sqrt{|\Delta_g|\frac{1}{2}Q^2})$$

with $-1 < \Delta_g < 0 \Rightarrow$ exponentially too small...

2. Very singular for purely electric states

3. Predicts duality noninvariant results

4. Using only monodromy invariance of $\Omega(p, q)$ in the large radius limit one can formally evaluate $Z_{bh} = \sum_q \Omega(p, q)e^{-\pi q\phi}$. The ϕ^A dependence is via a theta function - and does not reproduce the topological string answer.

Elaboration

(2): For $p^I = 0$, $\mathcal{F}^{\rm pert} = 0$ and $t^A = \phi^A/\phi^0 \in \mathbb{R}$.

$\Rightarrow$ automorphic functions on the real axis.

Example: $K3 \times T^2$: $|\Psi_{top}|^2 = |\Delta(\tau = \frac{\phi^1}{\phi^0})|^{-2}$:

$$\int |\Psi_{top}|^2 e^{\pi q \phi} = \delta(\vec{q}) \int d\phi^0 d\phi^1 \, \frac{e^{\pi q_0 \phi^0 + \pi q_1 \phi^1}}{|\Delta(\tau = \frac{\phi^1}{\phi^0})|^2}$$

(3): In fact, such states are related to p^1 $D4$ plus q_0 $D0$ by a duality transformation! $\Rightarrow$ OSV formula breaks duality invariance.

Conclusions

$$\boxed{\Omega(p,q) \overset{?}{=} \int d\phi \, |\Psi_{top}(p+i\phi)|^2 e^{\pi q\cdot\phi}}$$

Sometimes it works, and sometimes it doesn't.

But when it works, it works very well!

Because of the discrepancies it seems unlikely that it will apply as an exact formula valid for all (p,q).

This still leaves room for a nontrivial conjecture:

Perhaps it is only to be viewed as a statement about asymptotic expansions for large charges valid in regions of moduli space for BPS states with a black hole interpretation.

Note $\Omega_{abs} \sim \Omega_2$ in successful cases

Other lessons:

1. Ψ_{top} might not be in the Hilbert space for quantization of $H^3(\mathcal{X}, \mathbb{R})$.

2. The $\Omega(Q)$ are subtle arithmetic functions of Q and can change dramatically under small changes in "direction" of Q. Only "course-grained" versions of $\Omega(Q)$ have well-defined asymptotics.