Branes & Interfaces For 2D Landau-Ginzburg Models With Twisted Masses

Hamburg BPS States

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VI. SECOND CATEGORIFICATION: INTERFACES

I. MOTIVATION

$\Omega(\gamma)$ – much discussed

Categorification $\Omega(\gamma) \underset{\chi}{\overset{CAT}{\rightleftarrows}}$ Chain cplx

Done correctly: Captures more physical info.

Exple: Gaiotto, Moore, Witten (2015):

1+1 dim'l massive QFT w/ $U(1)_R$ charges, finite # of vacua $i, j, k \ldots$ "in general position":

- L_∞ algebraic structure on BPS states
- A_∞ Category of branes (related to Fukaya-Seidel for LG case)
- A_∞ 2-category of interfaces + Categorified ||-transport of cplx flat connections on Riemann surfaces

Question: Generalize to 4d theory?

Attempted by Dimofte - Gaiotto - Moore but set aside.

Revisiting question w/ AHSAN KHAN:
2d LG models w/ Twisted masses.
This incorporates many novel math features of the 4d problem.
(Special case of 2d4d systems discussed by P. Longhi yesterday.)

Work is in progress. If successful it should have numerous applications.

II. LG WITH TWISTED MASSES

(X, ω) Kähler manifold

$$\mathcal{P} := \left\{ \alpha \in \Omega^{1,0}(X),\ \bar\partial \alpha = 0, \quad \begin{array}{l} \text{isolated zeroes : nondegenerate } D\alpha|_{\phi_i} \text{ invtble} \\ \phi_i \end{array} \right\}$$

This ~~Data~~ $\Longrightarrow$ 1+1 dim'l QFT $LG(X, \alpha)$

Fields $\phi : \mathbb{M}^{1,1} \longrightarrow X$ + Fermi

Action $= \int \langle d\phi, *d\phi \rangle - \|\alpha\|^2 +$ Fermi

Math viewpoint : Morse theory on

$$\mathcal{H} = \{ \phi : \mathbb{R} \longrightarrow X \}$$

with Morse 1-form δh :

$$\begin{array}{ccc} \mathbb{R} \times \mathcal{X} & \xrightarrow{ev} & X \\ \downarrow p & & \\ \mathcal{X} & & \end{array}$$

$$\delta h = - p_* \left[ev^*(\omega) - \mathrm{Re}\left(\zeta\, ev^*(\alpha)\right) \wedge dx \right]$$

$$= - \int_{\mathbb{R}} \left[\omega_{\mu\nu}(\phi) \frac{d\phi^\mu}{dx} \delta\phi^\nu - \mathrm{Re}\left(\zeta \alpha_\mu(\phi) \delta\phi^\mu\right) dx \right]$$

ζ: phase (as in spectral networks) Could be absorbed into α but will be crucial when we replace $\mathbb{R} \longrightarrow [0, \infty)$

Zeroes(δh): $\quad \dfrac{d\phi}{dx} = \mathrm{HamFlow}\left(\mathrm{Re}(\zeta^{-1}\alpha)\right)$

Local halo coord's $\quad \underbrace{\dfrac{d\phi^I}{dx} = \zeta\, g^{I\bar{J}}\, \overline{\alpha_J(\phi)}}_{\text{"}\zeta\text{-soliton equation"}}$

B.C.'s $\quad \phi(x) \xrightarrow[x \to \pm\infty]{} \phi_i, \phi_j \in \mathrm{Zer}(\alpha)$

Physics is very different depending on whether α is exact or not.

$[\alpha] \in H^{1,0}(X)$ If $b_1(X) \neq 0$ can have $[\alpha] \neq 0 \Longleftrightarrow$ "twisted mass"

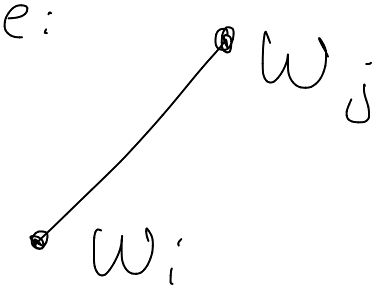
(A) $\alpha = dW$ exact: Very well-known story

$$\phi_i \xrightarrow[x]{\qquad\qquad} \phi_j \qquad i = j \Rightarrow \phi(x) = \phi_i \text{ const.}$$

(grad flow $(\mathrm{Im}\, \zeta^{-1} W)$)

$i \neq j$: Sol$^{\underline{n}}$ only exists for $\zeta = \zeta_{ij} = \dfrac{W_{ij}}{|W_{ij}|}$

projects to W-plane:



Important for usual story

- Finite # vacua
- W_i in "general position": No 3 collinear

(B) α NOT EXACT.

Pass to Abelian cover & work equivariantly

$\exists$ cover $\pi : \hat{X} \longrightarrow X$

$\hat{\alpha} = \pi^*(\alpha) = d\hat{W}$ exact

Deck group $= \Gamma =$ free Abelian

and : $\hat{\phi} \longrightarrow \gamma \cdot \hat{\phi}$ free action

Γ acts freely on set of vacua $\mathbb{V} := \{\phi_a\}_a$

write $a \longrightarrow a + \gamma$

$\Longrightarrow$ ∞ # vacua.

Moreover $\hat{W}_{a+\gamma} - \hat{W}_a = Z_\gamma = \oint_{\text{cycle}(\gamma)} \alpha$

$\Longrightarrow$
- ∞ly many vacua
- generically collinear critical values

Example: Mirror of free chiral w/ twisted mass

$$\hat{X} = \mathbb{C} \xrightarrow{\ \pi\ } X = \mathbb{C}^* \qquad \Gamma \cong \mathbb{Z}$$

$$\hat{\phi} \longmapsto \phi = e^{\hat{\phi}}$$

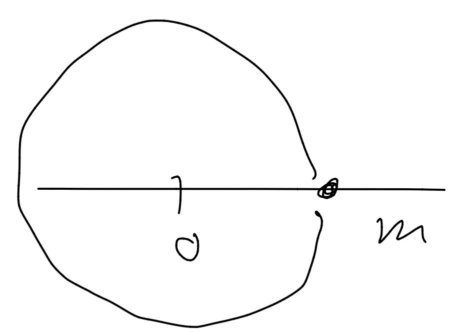
$$\hat{\alpha} = d\big(\underbrace{m\hat{\phi} - e^{\hat{\phi}}}_{\hat{W}}\big) \longleftarrow \alpha = \left(\frac{m}{\phi} - 1\right) d\phi$$

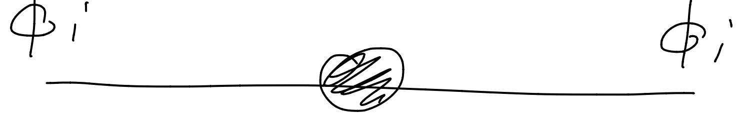
$$\hat{\phi}_k = \log m + 2\pi i k \qquad \phi_{cr} = m, \ i = 1$$
$$k \in \mathbb{Z}$$

$$\hat{\mathbb{V}} \longrightarrow \mathbb{V} = \hat{\mathbb{V}}/\Gamma = \{\phi_{cr}\}$$

$$\hat{W}_k = m \log m + 2\pi i k m$$

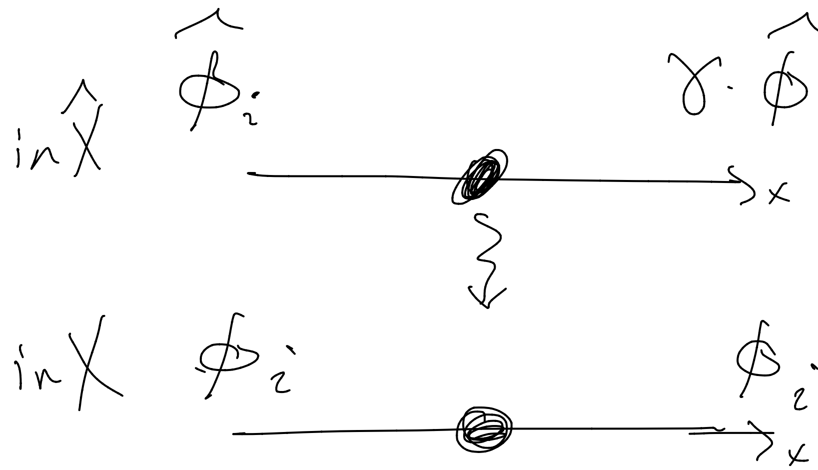
Also Periodic soliton





winding ± 1 times

In general: $\hat{\alpha} = d\hat{W}$ on $\hat{X}$



$$\zeta \parallel Z_\gamma$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\hat{\phi}"] \arrow[dr] & \hat{X} \arrow[d, "\pi"] \\
 & X
\end{tikzcd}$$

Has nontrivial flavor/winding charge

$$\int_{\mathbb{R}} \phi^*(\alpha) \quad \text{wrt} \quad J^\mu = \epsilon^{\mu\nu}\alpha_I(\phi)\partial_\nu\phi^I$$

MSW Complex (Morse-Smale-Witten)

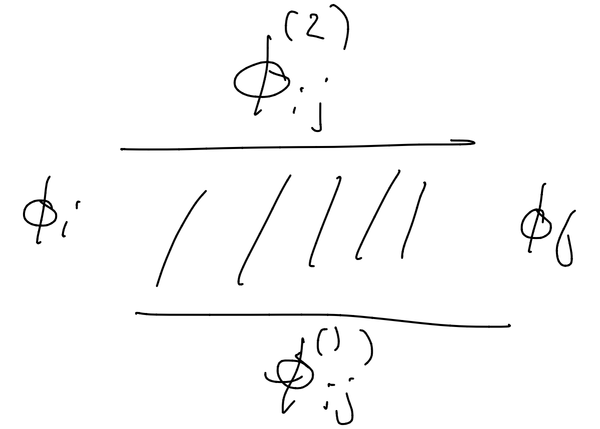
$\alpha = dW$ $\qquad \phi_i \quad \underline{\qquad\qquad} \quad \phi_j$

$$R_{ij}(X,\alpha) = \bigoplus_{\delta h(\phi_{ij})=0} \mathbb{Z} \cdot \Psi_{ij}$$

grading = Fermi # F = η (first order var. of ζ_{ij} sol eq.)

diffl:

d_{MSW}:



count ζ_{ij} - instantons (w/ signs)

ζ - inst. eq:

$$(\partial_x + i\partial_\tau)\phi^I = \bar\partial\phi^I = \zeta \cdot g^{I\bar J}\, \overline{\alpha_J(\phi)}$$

Index $\quad \mu_{ij}(X,\alpha) = \chi(R_{ij}) = \mathrm{Tr}_{R_{ij}}\, e^{i\pi F}$

$$= \mathcal{L}_i(\zeta_{ij} e^{-i\epsilon}) \circ \mathcal{L}_j(\zeta_{ij} e^{+i\epsilon}) \qquad \epsilon > 0$$

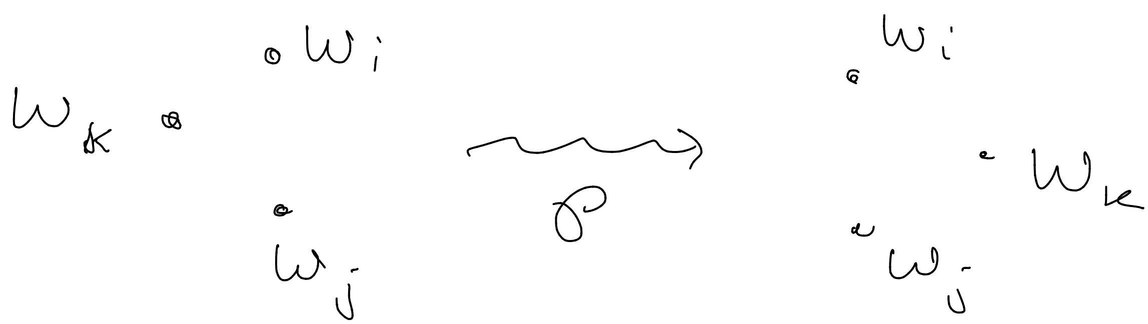
$$\mathcal{L}_i(\zeta) := \text{Left Lefshetz Thimble} \subset X$$

$$= \left\{ \phi \;\middle|\; \phi \xrightarrow[x \to -\infty]{\phi(x)} \phi_i \ \text{under } \zeta\text{-soliton flow} \right\}$$

<u>Remarks:</u>

1. W.C. $\mu_{ij}(X, \alpha)$ piecewise constant in α

$$\wp \supset W_{ijk} := \{ W \mid W_i \; W_j \; W_k \ \text{collinear} \}$$



$$\mu_{ik} \longrightarrow \mu_{ik} \pm \mu_{ij}\mu_{jk}$$

<u>Categorification Q:</u> R_{ij} Cat's μ_{ij}

What is cat. of WCF? How do R_{ij} change?

For $\{W_i\}$ in gen. position

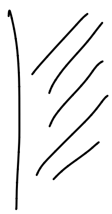
GMW (2015) had an answer. We will explain a litte more later.

III. PICARD-LEFSHETZ & SPECTRUM GENERATOR

Now consider ½ plane + branes preserving

$$\mathcal{Q}(\zeta) = \overline{Q}_+ + \zeta\, Q_-$$

(Rmk: Euclidean boost equiv. to rotation of ζ)

$\mathfrak{B}$ 

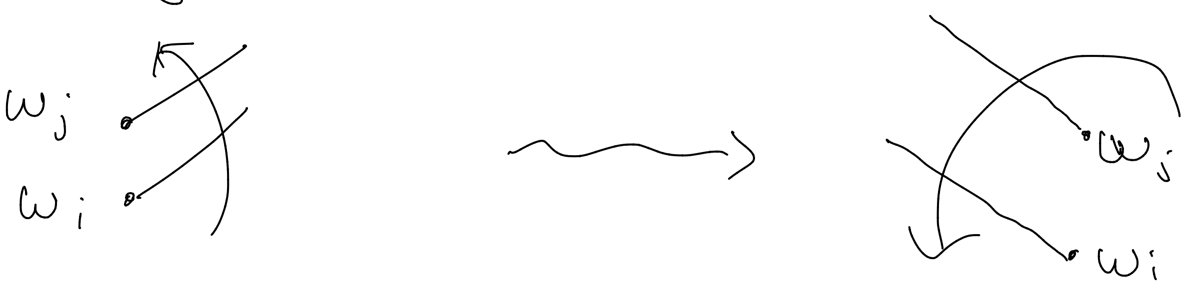
$$\mathfrak{B} \sim \Big(\overset{\text{Lagrangian}}{\underset{(X,\omega)}{\mathcal{L}}} \;,\; \underbrace{E \to \mathcal{L}}_{\text{Chan-Paton}} \Big)$$

with $\mathcal{L} \subset (X,\omega)$.

Fund. branes for $\alpha = dW$ are $\mathcal{L}_i(\zeta)$, define homology classes

$$L_i(\zeta) = [\mathcal{L}_i(\zeta)] \in H_{1/2}(X, \mathrm{Re}(\zeta W) \to \infty)$$

Change as function of ζ



$\arg \zeta < \arg \zeta_{ij}$ $\qquad\qquad$ $\arg \zeta > \arg \zeta_{ij}$

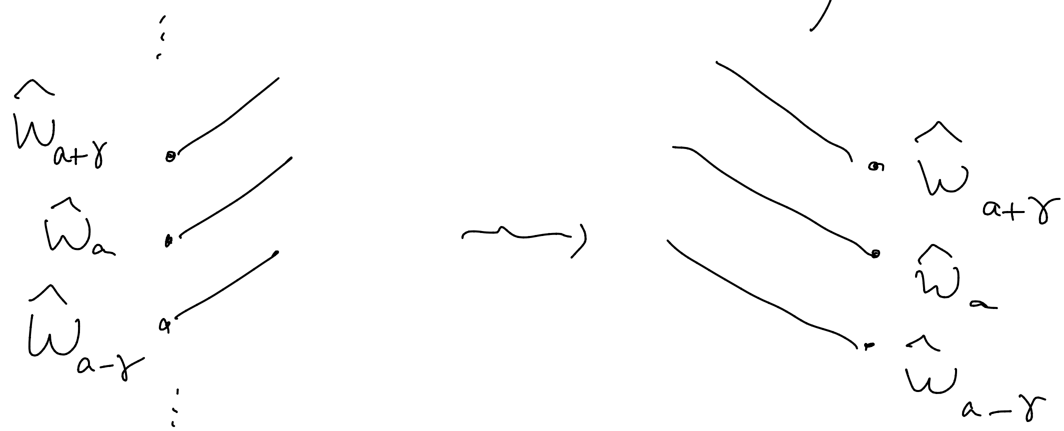
PL formula $\begin{pmatrix} L_1(\zeta) \\ \vdots \\ \vdots \\ L_n(\zeta) \end{pmatrix}$ ordered by $\mathrm{Im}(\zeta W)$

$$L(\zeta_{ij} e^{i\epsilon}) = S_{ij}^{\mu_{ij}} L(\zeta_{ij} e^{-i\epsilon}) .$$

$$S_{ij} = \mathbb{1} + e_{ij}$$

Can interpret as special kind of framed wall crossing: "S-wall-crossing."

When α not exact must pass to $\hat{X}$:



$\arg \zeta < \arg Z_\gamma$ $\qquad\qquad$ $\arg \zeta > \arg Z_\gamma$

Now $H_{1/2}(\hat{X}, \mathrm{Re}(\zeta^{-1}\hat{W}) \to \infty)$ is <u>Γ-module</u>

For each Γ orbit choose representative a_0 and <u>basis</u> $\{L_{a_0}(\zeta)\}_{a_0=1}^{a_0=\nu}$ $\qquad \nu := |\hat{V}/\Gamma| < \infty$

<u>as</u> $\mathbb{Z}[\Gamma]$-module. Define

$$A_\gamma := \sum_{a_0} \mu_{a_0, a_0+\gamma}\, e_{a_0 a_0} \qquad K_\gamma := (1 - \overset{\text{Deck trn}\ \downarrow}{T_\gamma})^{-1}$$

$$L(\zeta_\gamma e^{i\epsilon}) = \prod_{n=1}^{\infty} K_{n\gamma}^{A_{n\gamma}} \, L(\zeta_\gamma e^{-i\epsilon})$$

(γ primitive) : prove by carefully intersecting Lef. Thimbles

<u>Def:</u> $\hat{\mu}_{a_0,b_0}(\zeta) := \sum_{\gamma \in \Gamma} L_{a_0}(\zeta) \cap L_{b_0+\gamma}(\zeta e^{i\epsilon}) T_\gamma \in \mathbb{Z}[\Gamma]$

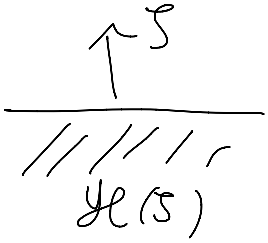
$$S(\zeta) := \sum_{a_0,b_0} \hat{\mu}_{a_0 b_0}(\zeta)\, e_{a_0 b_0} \in GL(\nu, \mathbb{Z}[\Gamma])$$

Following PL turns systematically rotating ζ_{ab} to ζ shows that $S(\zeta)$ factorizes:

$$S(\zeta) =: \overrightarrow{\prod} S_{ab}^{\mu_{ab}} K_\gamma^{A_\gamma} :$$

product over (a,b) : $Z_{ab} = \hat{W}_a - \hat{W}_b \in \mathcal{H}(\zeta)$

γ : $Z_\gamma \in \mathcal{H}(\zeta)$



$$S_{ab}^{\mu_{ab}} := \left(1 - T_\gamma e_{a_0 b_0}\right)^{\mu_{a_0, b_0+\gamma}}$$

WCF: $S(\zeta)$ = "Spectrum generator" invt

Example: Mirror of $\mathbb{CP}^1$ with twisted mass

$$\hat{X} = \mathbb{C} \xrightarrow{\pi} X = \mathbb{C}^* \qquad \Gamma = \langle T \rangle \cong \mathbb{Z}$$

$$\hat{\phi} \longmapsto \phi = e^{\hat{\phi}}$$

a generator of deck group

$$\hat{\alpha} = d\hat{W} \xleftarrow{\pi^*} \alpha = \left(\frac{t}{\phi^2} + \frac{m}{\phi} + t\right) d\phi$$

$$\hat{W} = m\hat{\phi} + t\left(e^{\hat{\phi}} + e^{-\hat{\phi}}\right)$$

$$\left|\frac{m}{t}\right| < 1 \qquad S = \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\left|\frac{m}{t}\right| > 1$$

$$S = \prod_{n=0}^{\infty} \begin{pmatrix} 1 & 0 \\ T^n & 1 \end{pmatrix} \begin{pmatrix} 1-T & \\ & (1-T)^{-1} \end{pmatrix} \prod_{n=\infty}^{1} \begin{pmatrix} 1 & -T^n \\ 0 & 1 \end{pmatrix}$$

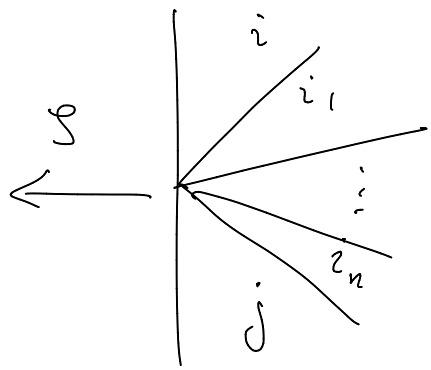
These are equal and we're going to categorify that statement.

IV. FIRST CATEGORIFICATION

When $\alpha = dW$ we only have S-factors and

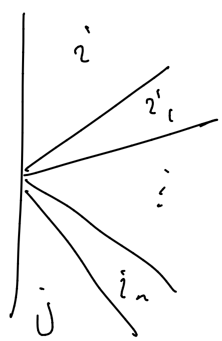
$$\hat{\mu}_{ij} = \mu_{ij} + \mu_{ii_1}\mu_{i_1 j} + \cdots$$

Corresponding Central charges are phase-ordered



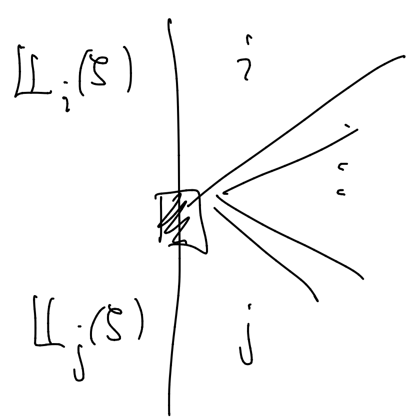
$\mu_{ij} \longrightarrow R_{ij}$ so

Consider



$$R_{ii_1} \otimes R_{i_1 i_2} \otimes \cdots \otimes R_{i_n j}$$

Interpret as v.s. of b.c.-changing operators



$$\hat{R}_{ij} := \bigoplus_{\substack{\frac{1}{2}\text{-plane} \\ \text{fans } \mathcal{F}}} R_{\mathcal{F}}$$

$\Rightarrow$ Category $Th(X, \alpha, \zeta)$ with objects $\mathbb{L}_i(\zeta)$

$$Hop(\mathbb{L}_i(\zeta), \mathbb{L}_j(\zeta)) := \hat{R}_{ij}$$

Composition of morphisms : $\hat{R}_{ij} \otimes \hat{R}_{jk} = \hat{R}_{ik}$

When phase-ordered

Note: $\hat{R}_{ij}$ inherits $\exists$ diff'l from d_{MSW} on R_{ij}

$\Rightarrow Th(X, \alpha, \zeta)$ is a dg category

$\boxed{\text{Surprise!!}}$ Wrong categorification, even though the indices are right.

Reason $H^*(R_{ij}, d_{MSW})$ jump $\Rightarrow H^*(\hat{R}_{ij}, d_{MSW})$ jump

$$\Rightarrow Th(X, \alpha_1, \zeta_1) \not\cong Th(X, \alpha_2, \zeta_2)$$

when α_1, α_2 sep. by $\mathcal{W}_{ijk}$.

Solution (GMW reinterpreted): If you modify d by ζ-instanton effects

$$d_{MSW} \longrightarrow d(\alpha) \quad \text{then:}$$

$$Th(X, \alpha_1, \zeta_1) \simeq Th(X, \alpha_2, \zeta_2)$$

Price: $Th(X, \alpha, \mathfrak{I})$ is an A_∞ category and this is an A_∞ equivalence — But all multiplications are computable in terms of $\mathfrak{I}$-instantons

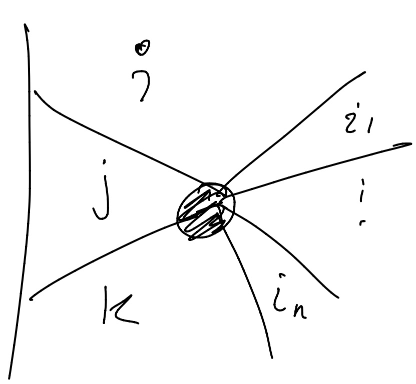
Put differently: Consider matrices of complexes

$$\mathcal{E}_{ij}(X,\alpha) = \mathbb{Z}\mathbb{1} + R_{ij}\, e_{ij}$$

$$\mathcal{E}_{\mathcal{H}(\mathfrak{I})}(X,\alpha) = \overrightarrow{\bigotimes_{z_{ij}\in\mathcal{H}(\mathfrak{I})}} \mathcal{E}_{ij}(X,\alpha)$$

As we'll argue later: This is invt on $\wp$ as long as no occupied rays enter/leave $\mathcal{H}(\mathfrak{I})$

Above uses $d(\alpha)$ and $V_1 \otimes_\alpha V_2$ is defined by considering pictures like



Pictures made precise by the "web formalism"

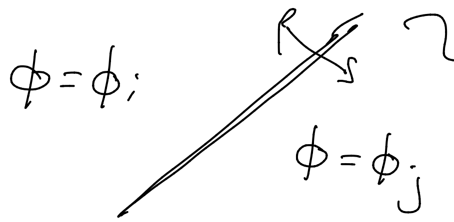
(and with twisted masses we need to generalize the web formalism)

V. ζ-INSTANTONS & L_∞-ALGEBRAS

- A ζ_{ij} soliton ϕ_{ij} gives a τ-independent solution of the ζ_{ij}-instanton equation.
- A rotation (= "Euclidean boost") in the (x,τ) plane rotates ζ:

$$\bar\partial \phi^I = \zeta \, g^{I\bar J} \, \overline{\alpha_J}$$

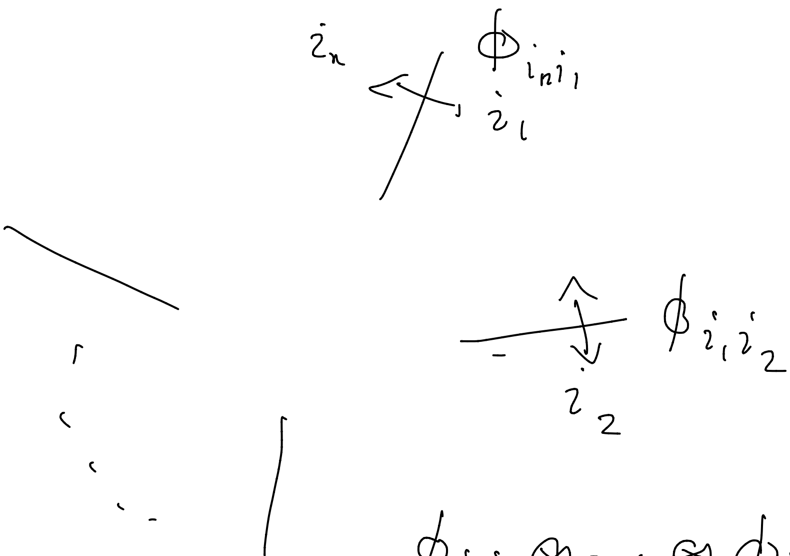
- A ζ_{ij} soliton can be rotated to give a solution of the ζ-instanton equation



$\rightsquigarrow$ slope $\parallel \zeta^{-1} z_{ij}$

"ζ-boosted soliton" soln to ζ-instanton equation

- Use this to define "fan boundary conditions @ ∞"



$$\phi_{i_1 i_2} \otimes \cdots \otimes \phi_{i_n i_1} \in R_{\mathcal{I}} = R_{i_1 i_2} \otimes \cdots \otimes R_{i_n i_1}$$

Path integral with these b.c.'s w/ action $LG(X,\alpha)$ defines a # ∴ path integral defines a vector in $(R^{int})^\vee$

$$R^{int} := \bigoplus_{\text{cyclic } \mathcal{F}} R_{\mathcal{F}}$$

Systematic discussion in GMW ⟹ this is an L_∞-algebra, and the vector defined by the path integral is a Maurer-Cartan element.

The argument

Uses the "web ~~formalism~~" – once again: That is what I'm generalizing with A.K.

VI. INTERFACES & "SECOND CATEGORIFICATION"

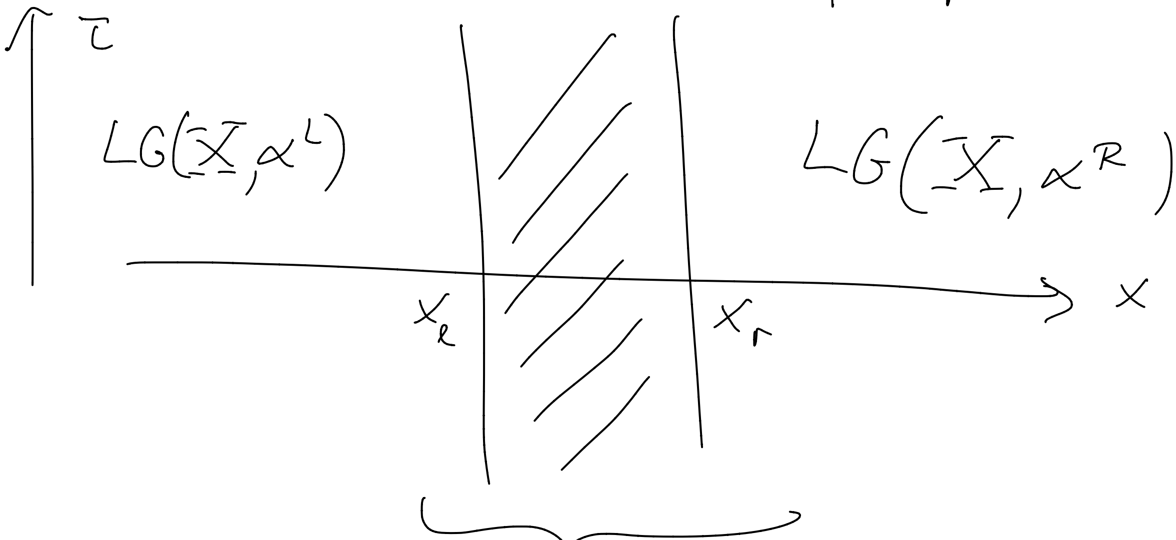
Now let $\alpha(\phi, x)$ also depend on spatial position x

$$\alpha \in \Omega^{1,0}(X \times \mathbb{R}) \quad \text{i.e.} \quad \alpha = \alpha_I(\phi, x)\, d\phi^I$$

$$\bar{\partial}_X \alpha = 0$$

The Morse 1-form δh on $\mathcal{X}$ still makes sense

<u>Physical Meaning</u>: Suppose $\partial_x \alpha$ has cpt support



Interface / Domain Wall / Janus

$\delta h = 0$: Forced ζ-soliton eq:

$$\frac{d\phi^I}{dx} = \zeta\, g^{I\bar{J}}\, \overline{\alpha_J(\phi, x)}$$

Note ζ arb. Now has solutions, in general.

$\wp : x \longmapsto \alpha(\cdot , x)$ path in $\mathcal{P}$

Form a matrix of complexes:

$$\mathcal{E}[\wp] = \sum_{i,j'} \overbrace{R[\wp]_{ij'}}^{\text{MSW cplx}} e_{ij'}$$

Case $\alpha = dW$ and $\wp$ crosses S-wall:

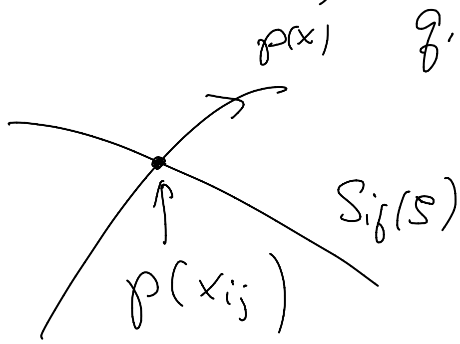
$$S_{ij}(\zeta) := \left\{ \alpha \;\middle|\; \begin{array}{l} W(\phi_i) - W(\phi_j) \parallel \zeta \\ \underline{\text{and}} \;\; \mu_{ij}(X, \alpha) \neq 0 \end{array} \right\}$$

(as in spectral networks!)

Categorified S-wall crossing (categorifies PL)

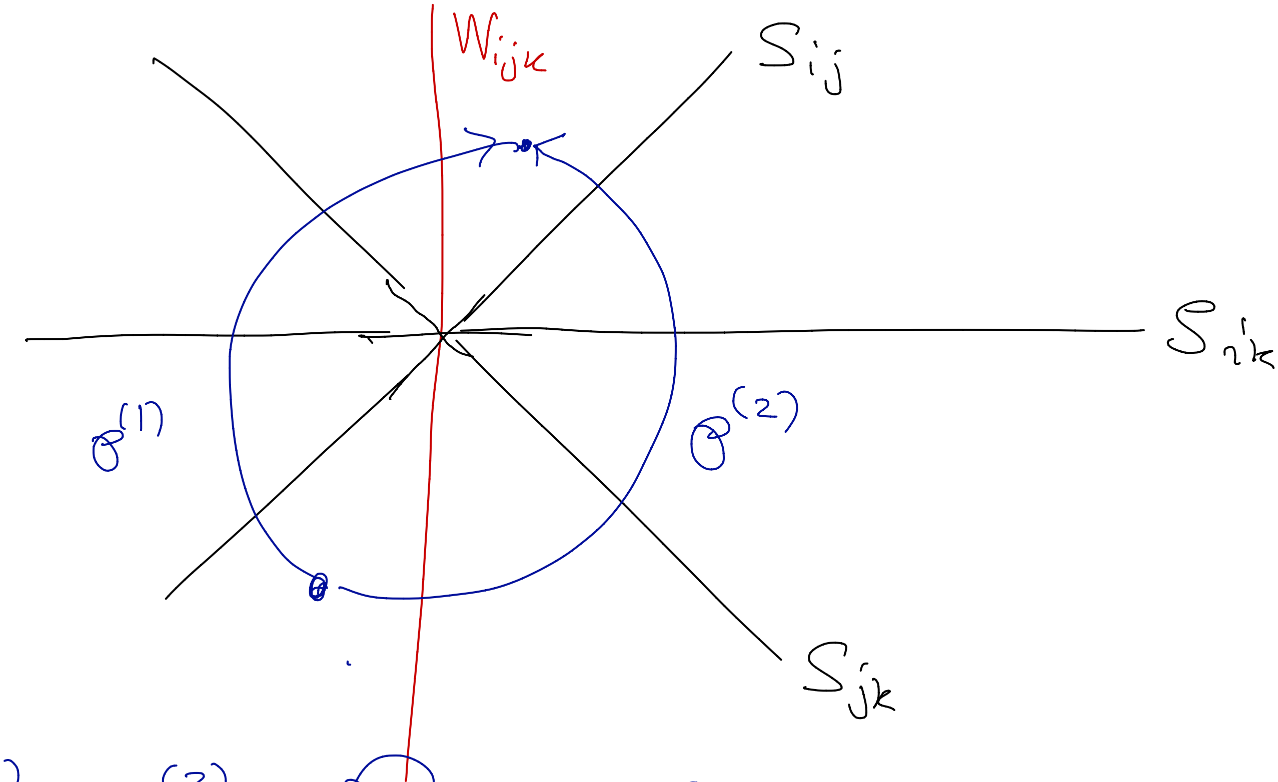
$$\wp^s(x) = \begin{cases} \wp(x) & x \leq s \\ \wp(s) & x \geq s \end{cases}$$

$$\mathcal{E}\left(\wp^{x_{ij}+\varepsilon}\right) \underset{q.i.}{\simeq} \mathcal{E}\left(\wp^{x_{ij}-\varepsilon}\right) \otimes \mathcal{E}_{ij}[\alpha(\cdot , x_{ij})]$$



$$\mathcal{E}_{ij}(\alpha) = \mathbb{1} + R_{ij}[\alpha] e_{ij}$$

$\Longrightarrow$ First categ. of CVWCF by a standard argument:



$\wp^{(1)} \underset{\text{h.e.}}{\sim} \wp^{(2)}$ in $\wp$ $\Longrightarrow$ $\mathcal{E}[\wp^{(1)}] \underset{q.i.}{\sim} \mathcal{E}[\wp^{(2)}]$

$\Longrightarrow$

$$\left(\mathcal{E}_{ij}\,\mathcal{E}_{ik}\,\mathcal{E}_{jk}\right)^{L} \underset{q.i.}{\sim} \left(\mathcal{E}_{jk}\,\mathcal{E}_{ik}\,\mathcal{E}_{ij}\right)^{R}$$

Taking χ $\Longrightarrow$ CVWCF.

$$\chi(\mathcal{E}_{ij}) = \left(\mathbb{1} + e_{ij}\right)^{\mu_{ij}}$$

But we can do much better:

Using the web formalism we construct an A_∞ category of interfaces:

$$\mathfrak{I}(X, \alpha_1 ; X, \alpha_2)$$

Together with A_∞ bifunctor:

$X_1\,\alpha_1 \;\Big|\; X_2\,\alpha_2 \;\Big|\; X_3\,\alpha_3$

$$\mathcal{I}(1,2)\times\mathcal{I}(2,3)\longrightarrow\mathcal{I}(1,3)$$

Given a path $\wp : \alpha_1 \rightsquigarrow \alpha_2$ get an object $I[\wp]$ in $\mathcal{I}(1,2)$ such that:

1. Homotopy class of $I[\wp]$ only depends on homotopy class of $\wp$.
2. $I(\wp_1)\circ I(\wp_2) \sim I(\wp_1\circ\wp_2)$

Cat of Parallel transport

3. $\exists$ S-wall interfaces $\sigma_{ij}^{\pm}$ s.t.

$$I(\wp^{x_{ij}+\varepsilon}) \underset{h.e.}{\sim} I(\wp^{x_{ij}-\varepsilon}) \boxtimes \sigma_{ij}^{\pm}$$

(2^{nd}) Categorified CVWCF:

$$(\sigma_{ij}\boxtimes\sigma_{ik}\boxtimes\sigma_{jk})^{L} \underset{h.e.}{\sim} (\sigma_{jk}\boxtimes\sigma_{ik}\boxtimes\sigma_{ij})^{R}$$

Generalization To Twisted Masses:

Requires construction of "K-wall interface"

What we do understand so far: Matrices of Complexes

in some examples - such as (mirror to) $\mathbb{CP}^1$

Strong Coupling

$$\mathcal{E}_{\mathcal{H}(\zeta)} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}[1]q \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

$d = d_{MSW}$

Weak Coupling

$$\mathcal{E}_{\mathcal{H}(\zeta)} = \bigotimes_{n=0}^{\infty} \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}q^n & \mathbb{Z} \end{pmatrix} \cdot \begin{pmatrix} \Lambda^*(q\mathbb{Z}[1]) & 0 \\ 0 & S^*(q\mathbb{Z}) \end{pmatrix} \cdot$$

$$\cdot \bigotimes_{n=\infty}^{1} \begin{pmatrix} \mathbb{Z} & \mathbb{Z}[1]q^n \\ 0 & \mathbb{Z} \end{pmatrix}$$

$d = d_{MSW}$ + Nontrivial instanton corrections

With a suitable differential these are q.i.

Note the appearance of <u>exterior</u> & <u>symmetric</u> algebras: It is clear from this, and other examples, that the K-wall interface will be closely related to Fock spaces and Koszul duality.