An Uncertainty Principle for Topological Sectors

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Introduction

Today I'll be talking about some interesting subtleties in the quantization of Maxwell's theory and some of its generalizations.

There are many different generalizations of Maxwell's theory...

Abelian gauge theories: fieldstrengths are differential forms $F \in \Omega^{\ell}(M)$.

"Generalized abelian gauge theories:" – the space of gauge-invariant field configurations "$\mathcal{A}/\mathcal{G}$" is a generalized differential cohomology group in the sense of Hopkins & Singer.

In particular "$\mathcal{A}/\mathcal{G}$" is an abelian group.

These kinds of theories arise naturally in supergravity and superstring theories, and indeed play a key role in the theory of D-branes and in claims of moduli stabilization in string theory made in the past few years.

Summary of the Results

1. Manifestly electric-magnetic dual formulation of the Hilbert space of generalized Maxwell theory.

2. The Hilbert space can be decomposed into electric and magnetic flux sectors, but the operators that measure electric and magnetic topological sectors don't commute and cannot be simultaneously diagonalized.

This is surprising and nontrivial!

3. Group theoretic approach to the theory of a self-dual field.

4. In particular: the K-theory class of a RR field cannot be measured!

Generalized Maxwell Theory

Begin with generalized Maxwell theory on a spacetime M with $\dim M = n$.

It has a fieldstrength $F \in \Omega^\ell(M)$

Action $S = \pi R^2 \int_M F * F$

If $M = X \times \mathbb{R}$ we have a Hilbert space $\mathcal{H}$.

$$dF = 0 \qquad \Rightarrow \qquad [F] \in H^\ell_{DR}(X)$$

more sophisticated…

Grade the Hilbert space by (the topological class of) magnetic flux:

$$\mathcal{H} = \oplus_m \mathcal{H}_m \qquad\qquad m \in H^\ell(X, \mathbb{Z}).$$

Motivating Question:

Electro-magnetic duality: An equivalent theory is based on a dual potential with $F_D \in \Omega^{n-\ell}(M)$ and

$$RR_D = \hbar$$

$\Rightarrow$ there must also be a grading by (topological class of) electric flux:

$$\mathcal{H} = \oplus_e \mathcal{H}_e \qquad\qquad e \in H^{n-\ell}(X, \mathbb{Z}),$$

Can we simultaneously decompose $\mathcal{H}$ into electric and magnetic flux sectors?

$$\mathcal{H} \stackrel{?}{=} \oplus_{e,m} \mathcal{H}_{e,m}.$$

``Isn't it obviously true?''

For the periodic 1+1 scalar this is just the decomposition into momentum + winding sectors!

Measure magnetic flux: $\int_{\Sigma_1} F$ where $\Sigma_1 \in Z_\ell(X)$

Measure electric flux: $\int_{\Sigma_2} *F$ where $\Sigma_2 \in Z_{n-\ell}(X)$

Canonical momentum conjugate to A is $\Pi \sim (*F)_X$.

$$[\int_{\Sigma_1} F, \int_{\Sigma_2} *F] \sim [\int_{\Sigma_1} F, \int_{\Sigma_2} \Pi] \tag{1}$$

$$= [\int_X \omega_1 F, \int_X \omega_2 \Pi]$$

$$= i\hbar \int_X \omega_1 d\omega_2 = 0 \tag{2}$$

where ω_i are closed forms Poincaré dual to Σ_i.

Fallacy in the argument

But! the above period integrals only measure the flux modulo torsion.

The topological sectors *e* and *m* are elements of *abelian groups*.

These abelian groups in general have nontrivial torsion subgroups.

Recall: A ``torsion subgroup'' of an abelian group *G* is the subgroup of elements of finite order:

$$0 \to H^{\ell}_{T}(X;\mathbb{Z}) \to H^{\ell}(X;\mathbb{Z}) \to H^{\ell}_{DR}(X)$$

The above discussion misses a very interesting uncertainty principle

What it is not:

1. We are not saying $$[E_i(x), B_j(y)] \neq 0$$

2. We are also not saying $$[\int_{\Sigma_1} F, \int_{\Sigma_2} *F] = L(C_1, C_2)$$

where $\partial\Sigma_1 = C_1$ $\partial\Sigma_2 = C_2$

3. Our uncertainty principle concerns ***torsion classes*** of topological sectors.

Differential Cohomology

Correct Mathematical Framework: Deligne-Cheeger-Simons Cohomology

To a manifold M and degree ℓ we associate $\check{H}^\ell(M)$.

Definition: By "generalized Maxwell theory" we mean a field theory such that the space of gauge inequivalent fields is $\check{H}^\ell(M)$ for some ℓ.

Simplest example: Periodic Scalar in 1+1 dimensions: F = dϕ

$$\check{H}^1(S^1) = Map(S^1, U(1)) = LU(1)$$

Next we want to get a picture of the space $\check{H}^\ell(M)$ in general

Structure of the Differential Cohomology Group

Fieldstrength exact sequence:

F

$$0 \to \overbrace{H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{H}^{\ell}(M) \xrightarrow{\text{fieldstrength}} \Omega^{\ell}_{\mathbb{Z}}(M) \to 0$$

Characteristic class exact sequence:

A

$$0 \to \underbrace{\Omega^{\ell-1}(M)/\Omega^{\ell-1}_{\mathbb{Z}}(M)}_{\text{Topologically trivial}} \to \check{H}^{\ell}(M) \xrightarrow{\text{char.class}} \underbrace{H^{\ell}(M;\mathbb{Z})}_{\text{Topological sector}} \to 0$$

<u>*Connected!*</u>

Structure of the Differential Cohomology Group

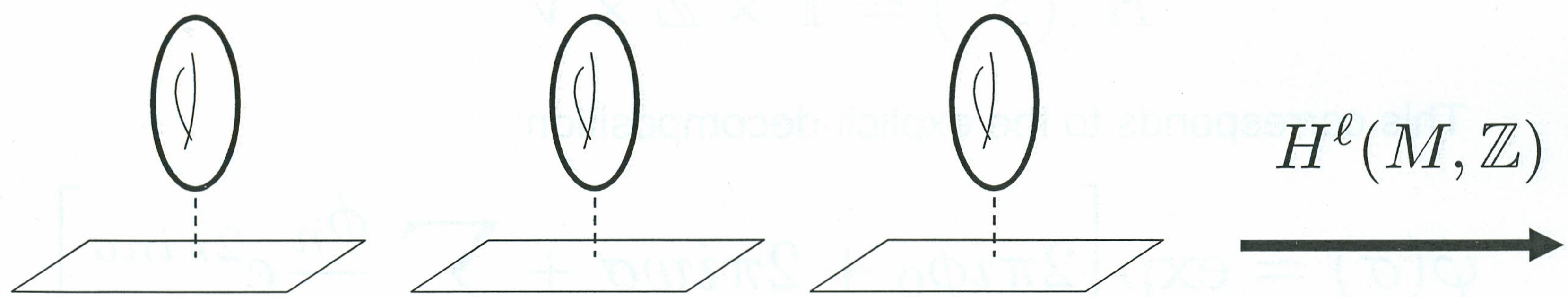
The space of differential characters has the form: $\check{H}^\ell = T \times \Gamma \times V$

T: Connected torus of topologically trivial flat fields:

$$\mathcal{W}^{\ell-1}(M) = H^{\ell-1}(M,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$$

Γ: Discrete (possibly infinite) abelian group of topological sectors: $H^\ell(M,\mathbb{Z})$.

V: Infinite-dimensional vector space of "oscillator modes." $V \cong \mathrm{Im} d^\dagger$.



Example: Loop Group of U(1)

Configuration space of a periodic scalar field on a circle:

$$\check{H}^1(S^1) = Map(S^1, U(1)) = LU(1)$$

Topological class = Winding number: $w \in H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$

Flat fields = Torus $\mathbb{T}$ of constant maps: $H^0(S^1, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$

Vector Space: $V = \Omega^0/\mathbb{R}$ Loops admitting a logarithm.

$$\check{H}^1(S^1) = \mathbb{T} \times \mathbb{Z} \times V$$

This corresponds to the explicit decomposition:

$$\varphi(\sigma) = \exp\left[2\pi i \phi_0 + 2\pi i w \sigma + \sum_{n \neq 0} \frac{\phi_n}{n} e^{2\pi i n \sigma}\right]$$

The space of flat fields

$H^{\ell-1}(X, \mathbb{R}/\mathbb{Z})$ Is a compact abelian group

... it is not necessarily connected! $\cong [Finite\ group] \times [Torus]$

Connected component of identity: $\mathcal{W}^{\ell-1}(M) = H^{\ell-1}(M, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$

Group of components: $= H^{\ell}_{T}(M; \mathbb{Z})$

Example: $\ell = 2,\ M = S^3/\mathbb{Z}_k \quad H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong \mathbb{Z}_k$

$$H^1(M; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_k$$

Discrete Wilson lines, defining topologically nontrivial flat gauge fields.
Entirely determined by holonomy:

$$\chi_r(\gamma) = e^{2\pi i r/k} = \omega_k^r \qquad\qquad r \in \mathbf{Z}_k$$

Poincare-Pontryagin Duality

M is compact, oriented, dim(M) = n

There is a very subtle ***PERFECT PAIRING*** on differential cohomology:

$$\check{H}^{\ell}(M) \times \check{H}^{n+1-\ell}(M) \to \mathbb{R}/\mathbb{Z}$$

$$\langle [\check{A}_1], [\check{A}_2] \rangle := \int_M^{\check{H}} [\check{A}_1] * [\check{A}_2]$$

On topologically trivial fields:

$$\langle [A_1], [A_2] \rangle = \int_M A_1 dA_2 \mod \mathbb{Z}$$

Example:Cocycle of the Loop Group

Recall $\check{H}^1(S^1) = LU(1)$:

$$\varphi = \exp(2\pi i \phi) \qquad \phi : \mathbb{R} \to \mathbb{R}$$

$$\phi(s+1) = \phi(s) + w \qquad w \in \mathbb{Z} \text{ is the winding number.}$$

$$\langle \varphi^1, \varphi^2 \rangle = \int_0^1 \phi^1 \frac{d\phi^2}{ds} ds - w^1 \phi^2(0) \qquad \mathrm{mod}\ \mathbb{Z}$$

Note! This is (twice!) the cocycle which defines the basic central extension of $LU(1)$.

Hamiltonian Formulation of Generalized Maxwell Theory

Spacetime: $M = X \times \mathbb{R}$.
Generalized Maxwell fields: $[\check{A}] \in \check{H}^{\ell}(M)$.

$$S = \pi R^2 \int_M F * F$$

Hilbert space: $\mathcal{H} = L^2(\check{H}^{\ell}(X))$

This breaks manifest electric-magnetic duality.

There is a better way to characterize the Hilbert space.

But for the moment we use this formulation.

Defining Magnetic Flux Sectors

Return to our question:

Can we simultaneously decompose $\mathcal{H}$ into electric and magnetic flux sectors?

$$\mathcal{H} \stackrel{?}{=} \oplus_{e,m} \mathcal{H}_{e,m}.$$

For definiteness, choose the duality frame with $\mathcal{H} \cong L^2(\check{H}^\ell)$.

Definition: A state of definite topological class of magnetic flux has its wavefunction supported on the component labelled by

$$m \in H^\ell(X, \mathbb{Z}).$$

Electric flux can be defined similarly in the dual frame, $\mathcal{H} \cong L^2(\check{H}^{n-\ell})$ but…

Defining Electric Flux Sectors

We need to understand more deeply the grading by electric flux in the polarization $L^2(\check{H}^\ell)$.

$$\text{Diagonalizing } (*F)_X \text{ means diagonalizing } \Pi,$$

$$\Pi \text{ is the generator of translations.}$$

Definition: A state of definite topological class of electric flux is an eigenstate under translation by *flat fields*

$$\forall \phi_f \in H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}), \quad \psi(\check{A} + \check{\phi}_f) = \exp\Big(2\pi i \int_X^H e\phi_f\Big)\psi(\check{A})$$

The topological classes of electric flux are labelled by $e \in H^{n-\ell}(X, \mathbb{Z})$.

Noncommuting Fluxes

So: the decomposition wrt electric flux is diagonalizing the group

of translations by flat fields: $H^{\ell-1}(X, \mathbb{R}/\mathbb{Z})$

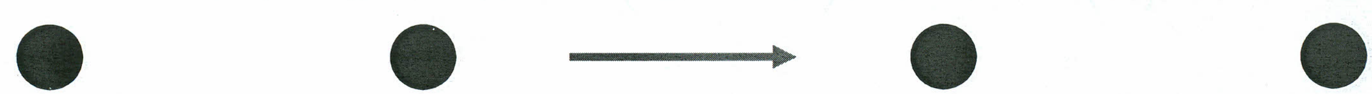
Suppose ψ is in a state of definite magnetic flux $m \in H^{\ell}(X, \mathbb{Z})$:

the support of the wavefunction ψ is in the topological sector $\check{H}^{\ell}(X)_m$.

Such a state cannot be in an eigenstate of translations by flat characters, if there are any flat, but topologically nontrivial fields.

Translation by such a nontrivial flat field must translate the support of ψ to a different magnetic flux sector.

Therefore, one cannot in general measure both electric and magnetic flux.



Group Theoretic Approach

Let S be any (locally compact) abelian group

$\mathcal{H} = L^2(S)$ is a representation of S: $\forall s_0 \in S$

$$(T_{s_0}\psi)(s) := \psi(s + s_0).$$

Let $\hat{S}$ be the Pontryagin dual group of characters of S

$\mathcal{H} = L^2(S)$ is also a representation of $\hat{S}$: $\forall \chi \in \hat{S}$

$$(M_\chi \psi)(s) := \chi(s)\psi(s)$$

But! $T_{s_0} M_\chi = \chi(s_0) M_\chi T_{s_0}.$

So $\mathcal{H} = L^2(S)$: is a representation of the Heisenberg group central extension:

$$1 \to U(1) \to \mathrm{Heis}(S \times \hat{S}) \to S \times \hat{S} \to 1$$

Heisenberg Groups

Theorem A Let G be a topological abelian group. Central extensions, $\tilde{G}$, of G by $U(1)$ are in one-one correspondence with continuous bimultiplicative maps $s : G \times G \to U(1)$ which are alternating (and hence skew).

1. s is *alternating*: $s(x,x) = 1$.
2. s is *skew* : $s(x,y) = s(y,x)^{-1}$.
3. s is *bimultiplicative*:

$$s(x_1 + x_2, y) = s(x_1, y)s(x_2, y) \quad \& \quad s(x, y_1 + y_2) = s(x, y_1)s(x, y_2)$$

If $x \in G$ lifts to $\tilde{x} \in \tilde{G}$ then $s(x,y) = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$

Definition: If s is nondegenerate then $\tilde{G}$ is a *Heisenberg group*.

Theorem B: (Stone-von Neuman theorem). If $\tilde{G}$ is a Heisenberg group then the unitary irrep of $\tilde{G}$ where $U(1)$ acts canonically is unique up to isomorphism.

Heisenberg group for generalized Maxwell theory

If $S = \check{H}^{\ell}(X)$, then PP duality $\Rightarrow \hat{S} = \check{H}^{n-\ell}(X)$:

$$\tilde{G} := \mathrm{Heis}\big(\check{H}^{\ell}(X) \times \check{H}^{n-\ell}(X)\big)$$

via the group commutator:

$$s\left(([\check{A}_1],[\check{A}_1^D]),([\check{A}_2],[\check{A}_2^D])\right) = \exp\left[2\pi i\left(\langle[\check{A}_2],[\check{A}_1^D]\rangle - \langle[\check{A}_1],[\check{A}_2^D]\rangle\right)\right].$$

The Hilbert space of the generalized Maxwell theory *is* the unique irrep of the Heisenberg group $\tilde{G}$

N.B! This formulation of the Hilbert space is *manifestly electric-magnetic dual*.

Flux Sectors from Group Theory

Electric flux sectors diagonalize the flat fields $H^{\ell-1}(X,\mathbb{R}/\mathbb{Z})$

Magnetic flux sectors diagonalize dual flat fields $H^{n-\ell-1}(X,\mathbb{R}/\mathbb{Z})$

These groups separately lift to commutative subgroups of $\tilde{G} := \mathrm{Heis}(\check{H}^{\ell}\times\check{H}^{n-\ell})$.

However they do not commute with each other!

$$\mathcal{U}_E(\eta_e) := \text{translation operator by } \eta_e \in H^{\ell-1}(X,\mathbb{R}/\mathbb{Z})$$

$$\mathcal{U}_M(\eta_m) := \text{translation operator by } \eta_m \in H^{n-\ell-1}(X,\mathbb{R}/\mathbb{Z})$$

$$[\mathcal{U}_e(\eta_e),\mathcal{U}_m(\eta_m)] = T(\eta_e,\eta_m) = \exp\left(2\pi i\int_X \eta_e\beta\eta_m\right)$$

T: torsion pairing, $\beta=$ Bockstein: $\beta(\eta_m)\in H^{n-\ell}_T(X,\mathbb{Z})$.

The New Uncertainty Relation

$$[\mathcal{U}_e(\eta_e), \mathcal{U}_m(\eta_m)] = T(\eta_e, \eta_m) = \exp\left(2\pi i \int_X \eta_e \beta \eta_m\right)$$

T torsion pairing, ***β*** = Bockstein.

Translations by $\mathcal{W}^{\ell-1}(X)$ and by $\mathcal{W}^{n-\ell-1}(X)$ commute

$$\Longrightarrow \qquad \mathcal{H} = \oplus_{\bar{e},\bar{m}} \mathcal{H}_{\bar{e},\bar{m}}.$$

However: The pairing does <u>*not*</u> commute on the subgroups of all flat fields.

It descends to the ``torsion pairing'' or ``link pairing'':

$$H^{\ell}_{T}(X) \times H^{n-\ell}_{T}(X) \to \mathbb{R}/\mathbb{Z}$$

This is a perfect pairing, so it is maximally noncommutative on torsion.

Maxwell theory on a Lens space

$$S^3/\mathbb{Z}_k \times \mathbb{R}$$

$H^1(L_k;\mathbb{R}/\mathbb{Z}) \cong H^2(L_k;\mathbb{Z}) = \mathbb{Z}_k$ is all torsion

Acting on the Hilbert space the flat fields generate a Heisenberg group extension

$$0 \to \mathbb{Z}_k \to \mathrm{Heis}(\mathbb{Z}_k \times \mathbb{Z}_k) \to \mathbb{Z}_k \times \mathbb{Z}_k \to 0$$

This has unique irrep ***P*** = clock operator, ***Q*** = shift operator

$$PQ = e^{2\pi i/k} QP$$

States of definite electric and magnetic flux $|e\rangle = \frac{1}{\sqrt{k}} \sum_m e^{2\pi i e m/k} |m\rangle$

This example already appeared in string theory in Gukov, Rangamani, and Witten, hep-th/9811048. They studied AdS5xS5/Z3 and in order to match nonperturbative states concluded that in the presence of a D3 brane one cannot simultaneously measure D1 and F1 number.

Remarks- Principles of Quantum Mechanics

The pairing of topologically nontrivial flat fields has no tunabale parameter. Therefore, this is a quantum effect which does not become small in the semiclassical or large volume limit.

In general, there are no clear rules for quantization of disconnected phase spaces. We are extending the standard rules of quantum mechanics by demanding electric-magnetic duality.

This becomes even more striking in the theory of the self-dual field.

An Experimental Test

Since our remark applies to Maxwell theory: Can we test it experimentally?

Discouraging fact: No region in $\mathbb{R}^3$ has torsion in its cohomology

With A. Kitaev and K. Walker we noted that using arrays of Josephson Junctions and a configuration called ``superconducting mirros,'' we can ``trick'' the Maxwell field into behaving as if it were in a space with torsion.

To exponentially good accuracy the groundstates of the electromagnetic field are an irreducible representation of $\mathrm{Heis}(\mathbb{Z}_n \times \mathbb{Z}_n)$

See arXiv:0706.3410 for more details.

Self-dual fields

Now suppose $\dim M = 4k+2$, and $\ell = 2k+1$.

We can impose a self-duality condition ***F* = *F**.

This is a very subtle quantum theory: Differential cohomology is the ideal tool.

For the non-self-dual field we represent $\mathrm{Heis}(\check{H}^{\ell}(X) \times \check{H}^{\ell}(X))$

Proposal: For the self-dual field we represent: $\mathrm{Heis}(\check{H}^{\ell}(X))$

Attempt to define this Heisenberg group via

$$s_{\mathrm{trial}}([\check{A}_1],[\check{A}_2]) = \exp 2\pi i \langle [\check{A}_1],[\check{A}_2] \rangle .$$

It is skew and and nondegenerate, but *not* alternating!

$$s_{\mathrm{trial}}([\check{A}],[\check{A}]) = (-1)^{\int_X \nu_{2k}\, m}$$

Gomi 2005

$\mathbb{Z}_2$ -graded Heisenberg groups

Theorem A': Skew bimultiplicative maps classify $\mathbb{Z}_2$ -graded Heisenberg groups.

$\mathbb{Z}_2$ grading in our case:

$$\epsilon([\check{A}]) = 0 \qquad \text{if} \qquad \int \nu_{2k}\, m = 0 \bmod 2$$

$$\epsilon([\check{A}]) = 1 \qquad \text{if} \qquad \int \nu_{2k}\, m = 1 \bmod 2$$

Theorem B': A $\mathbb{Z}_2$-graded Heisenberg group has a unqiue $\mathbb{Z}_2$ -graded irreducible representation.

This defines the Hilbert space of the self-dual field

Example: Self-dual scalar: $k = 0$. By bosonization $\psi = e^{i\phi}$.
The $\mathbb{Z}_2$-grading is just *fermion number*!

Relation to Nonselfdual Field

Remark: One can show that the nonself-dual field at a special radius, $R^2 = 2\hbar$, decomposes into

$$\mathcal{H}_{nsd} \cong \oplus_\alpha \mathcal{H}_{sd,\alpha} \otimes \mathcal{H}_{asd,\alpha}$$

The subscript α is a sum over "generalized spin structures" -

a torsor for 2-torsion points in $H^{2k+1}(X;\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$.

For the self dual scalar α labels R and NS sectors.

Thus, self-dual theory generalizes VOA theory

String Theory Applications

1. RR field of type II string theory is valued in differential K-theory $\check{K}(X)$

RR field is self dual, hence Hilbert space = unique irrep of $\mathrm{Heis}(\check{K}(X))$

$$\boxed{\text{Hence – the full K theory class is not measurable!}}$$

2. B-fields: String theory backgrounds with nontrivial discrete fluxes for both B_2 and B_6 do not exist

3. AdS/CFT Correspondence: Discrete symmetries of quiver gauge theories are in correspondence with $\mathrm{Heis}(H_T^*(X_5)))$

X_5 an Einstein-Sasaki 5-manifold

Open Problems

1. What happens when $\boldsymbol{X}$ is noncompact?

2. If one cannot measure the complete *K*-theory class of RR flux what about the D-brane charge?

a.) If no, we need to make an important conceptual revision of the standard picture of a D-brane

b.) If yes, then flux-sectors and D-brane charges are classified by different groups $\Longrightarrow$ tension with AdS/CFT and geometric transitions.

3. What is the physical meaning of the fermionic sectors in the RR Hilbert space?

4. How is this compatible with noncommutativity of 7-form Page charges in M-theory?