SPECIAL TOPICS SOLID STATE PHYSICS: 681. Fall 2019

Questions I. (Due Mon, Sept 30th.)

1. Consider the most general form of a two component Landau theory

$$f[\psi] = \frac{r}{2}(\psi_1^2 + \psi_2^2) + \frac{s}{2}(\psi_1^2 - \psi_2^2) + u(\psi_1^2 + \psi_2^2)^2 + u_2(\psi_1^4 - \psi_2^4) + u_3\psi_1^2\psi_2^2$$

where $\psi = \psi_1 + i\psi_2 = |\psi|e^{i\phi}$ is the complex order parameter.

   (a) Rewrite the free energy in terms of the amplitude $|\psi|$ and phase ϕ of the order parameter to demonstrating that if s, u_2 or u_3 are finite, the free energy is no longer gauge invariant, i.e $F[\psi] \neq F[\psi e^{i\alpha}]$.

   (b) Rewrite the free energy as a function of ψ and ψ^*.

   (c) If $s > 0$, what symmetry is broken when $r < 0$ is negative?

   (d) Write down the mean field equations for $s = 0$, $r < 0$.

   (e) Sketch the phase diagram in the (u_2, u_3) plane, trying to identify three phases.

2. Show that the action of $U(\phi) = e^{i\phi\hat{N}}$ on a coherent state, $|\psi\rangle$, $U(\phi)|\psi\rangle = |\tilde{\psi}\rangle$ uniformly shifts the phase of the order parameter by ϕ, i.e.

$$\hat{\psi}(x)|\tilde{\psi}\rangle = \psi(x)e^{i\phi}|\phi\rangle$$

so that $\tilde{\psi} = e^{i\phi}\psi$ and $|\tilde{\psi}\rangle = |\psi e^{i\phi}\rangle$. Show that this implies

$$-i\frac{d}{d\phi}|e^{i\phi}\psi\rangle = \hat{N}|e^{i\phi}\psi\rangle$$

3. Properties of a coherent state.

   (a) Show that a coherent state $|\alpha\rangle = e^{\alpha a^\dagger}|0\rangle$ can be expanded as a sum of Harmonic oscillator states $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$, as follows

$$|a\rangle = |0\rangle + \alpha|1\rangle + \ldots \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

   (b) Show that $\langle\alpha^*|\alpha\rangle = e^{|a|^2}$, so that a normalized coherent state is given by

$$|\alpha\rangle_N = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle$$

   (c) Show that the probabilty of being in a state with n particles is a Poisson distribution

$$p(n) = \frac{(\lambda)^n}{n!}e^{-|\lambda|}, \qquad \lambda = |\alpha|^2$$

   Note that a Poission distribution has equal mean and variance : $\langle\hat{N}\rangle = \langle\delta\hat{N}^2\rangle = \lambda$

(d) Show that when $\alpha = \sqrt{N_s}$, $\frac{\delta N^2}{N^2} = \frac{1}{N_s}$.

4. (a) Show that the quantum dynamics of a superfluid, in which the energy functional $H = H(N, \phi)$ is a function of particle number N and condensate phase ϕ must obey Hamiltonian dynamics

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{i}{\hbar}[N, H] = \frac{\partial H}{\partial \phi} \\ \frac{\partial \phi}{\partial t} &= \frac{i}{\hbar}[\phi, H] = -\frac{\partial H}{\partial N}. \end{aligned} \tag{1}$$

(Hint, use the fact that $N = i\frac{d}{d\phi}$, in a coherent state, and that equivalently $\phi = i\frac{d}{dN}$ in a state of definite N.) Compare your results with Hamiltons equations.

(b) Show that in equilibrium the superfluid has the following time dependence

$$\psi(x, t) = \psi(x, 0)e^{-i\mu t/\hbar} \tag{2}$$

(c) Josephson effect If two superfluid reservoirs with phases ϕ_1 and ϕ_2 are connected via a weak link, the energy between the two reservoirs must be a periodic function of the phases, and in its simplest form takes the Josephson form

$$H = -J\cos(\phi_2 - \phi_1)$$

Show that the current flowing from reservoir 1→2 is

$$I = \frac{\partial N_2}{\partial t} = -\frac{\partial N_1}{\partial t} = \frac{J}{\hbar}\sin(\phi_2 - \phi_1). \tag{3}$$

and that if the chemical potentials are fixed, the current will undergo oscillations according to

$$I(t) = \frac{J}{\hbar}\sin\left(\frac{\mu_1 - \mu_2}{\hbar}t + \phi\right) \tag{4}$$

This is the AC Josephson effect.