# Teaching the renormalization group[a]

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The renormalization group theory of second-order phase transitions is described in a form suitable for presentation as part of an undergraduate statistical physics course.

## INTRODUCTION

The problem of communicating exciting new developments in physics to undergraduate students is of great importance. There is a natural tendency for students to believe that they are a long way from the frontiers of science where discoveries are still being made. The presentation to undergraduates of important new discoveries can show them that they are nearer the frontier than they had imagined. The effect on students' interest and enthusiasm is especially marked when they find that the "great new discovery" that they had heard mentioned so many times without actually knowing what it was turns out to be rather simple, and something they might have thought of themselves given a little time and a better background. Students tend to think that the nature and manner of physics research changed abruptly at some time (possibly about 40 years ago). Before this time, physicists thought deeply and had sudden inspirations that enabled them to proceed by giant strides. Later on, however, physics became very technical and detailed. In this later period, which includes the present time, so students believe, progress could only be made after enormous study and effort because all of the simple things had been done. The realization that the new developments in physics are really not essentially different from the old dispels these ideas about the degeneration of our subject.

In this paper, we want to describe how an important recent development in theoretical physics, namely, the renormalization group, can be successfully incorporated into a junior-level course in statistical physics. We assume that the material is presented at the end of a one-semester course using a text such as Reif's *Statistical and Thermal Physics*.[1] A student thus already knows that:

(1) Substances have various phases which can be represented on a phase diagram in the pressure-temperature plane. Going from one phase to another is accompanied by discontinuous changes in entropy and volume.

(2) All thermodynamic quantities can be calculated from the partition function $Z$, defined by

$$Z = \sum_n e^{-E_n/k_B T}, \tag{1}$$

where $E_n$ is the $n$th energy state of the system, $k_B$ is Boltzmann's constant, and $T$ is the temperature.

(3) Ordinary phase transitions (i.e., first-order transitions) can be understood in a relatively simple way be treating the various phases as separate entities each with its own Gibbs free energy.[2]

## SECOND-ORDER PHASE TRANSITIONS

By way of introduction to the topic of second-order phase transitions some examples are useful to the student. Suitable choices include the following:

(a) Order-disorder transitions. Certain materials undergo a phase transition in which the specific heat $C$ has a singularity of the general form shown in Fig. 1(a). Close to the transition temperature $T_c$, $C$ is proportional to

$$|T - T_c|^{-\alpha}, \tag{2}$$

where $\alpha$ is typically around 0.1. There is no latent heat. The standard example of this type of transition is $\beta$-brass. This is an alloy of copper and zinc, in equal amounts, with a body-centered-cubic crystal structure. Above the critical temperature $T_c$ of 460 °C a copper atom has, on the average, as many copper nearest neighbors as zinc nearest neighbors. As the temperature is lowered below $T_c$, however, there develops an increasing probability for a copper atom to have more zinc nearest neighbors than would result from a completely random arrangement of copper and zinc atoms. This probability increases so that at temperatures much less than $T_c$ each copper atom has only zinc neighbors and each zinc atom has only copper neighbors. Thus one says that there is a transition from complete disorder above $T_c$ to a phase which has continuously increasing order as the temperature is lowered below $T_c$.

(b) Paramagnetic-ferromagnetic transition. Above $T_c$ the magnetic dipoles in a solid are randomly oriented, and so in the absence of an applied magnetic field there is no net magnetic moment. Below $T_c$ there is a partial alignment of the dipoles and a so-called "spontaneous magnetization" [Fig. 1(b)]. This magnetization **M** is a vector whose direction is not unique. Thus upon heating a magnet above $T_c$ and recooling the same magnitude of magnetization will occur but the direction may be different. It is found experimentally that for $T$ close to $T_c$

$$|\mathbf{M}| \propto (T_c - T)^\beta \tag{3}$$

for $T < T_c$. The critical exponent lies in the range 0.3–0.4. The specific heat also has a singularity proportional to

$$|T - T_c|^{-\alpha} \tag{4}$$

with $\alpha$ in most cases being close to zero.

These two examples show clearly that there are in nature more subtle transitions than those of the simple first-order variety. To understand how this can occur it is useful to consider a very simplified model, the Ising model, which nevertheless manages to retain enough of the essential features to be useful. One can choose to think of this model as a model of the magnetic transition, but it is also just as good a representation of the order-disorder transition. The model involves $N$ magnetic dipoles which we will call spins for short. The simplifications made are the following:

(1) Each spin can point only up or down. These directions are chosen to be the positive and negative $z$ directions. Thus
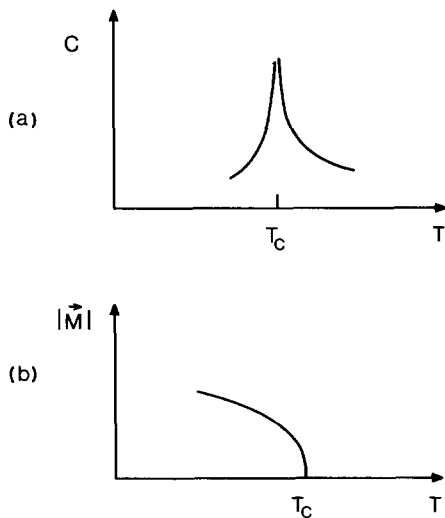
Fig. 1. Examples of singularities in thermodynamic functions at second-order phase transitions. (a) Specific heat $C$ at an order-disorder transition. (b) Magnitude $|\vec{M}|$ of the spontaneous magnetization at a magnetic phase transition.

one can describe the state of any spin by a number $\sigma$ which is $+1$ for up and $-1$ for down, and the state of the whole system is specified by giving the value of $\sigma$ for each of the $N$ spins.

(2) The spins are arranged in a simple-cubic lattice. Thus there is a spin at each of the positions

$$\mathbf{X}_{lmn} = l\hat{i} + m\hat{j} + n\hat{k}, \tag{5}$$

where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors parallel to the Cartesian axes, and $l$, $m$, and $n$ are integers. To be more precise this is the arrangement of spins in the three-dimensional cubic Ising model. In the square two-dimensional model the spins are located in a square array in the $x$-$y$ plane,[3] and in the one-dimensional model they are at equally spaced intervals along the $x$ axis.

(3) There is an interaction energy between spins that tends to align them. Each pair of nearest-neighbor spins that are parallel makes a contribution $-J$ to the total energy of the system, and each nearest-neighbor pair that are antiparallel gives a contribution $+J$. $J$ is a positive constant. Thus the total energy can be written

$$E = -J \sum_{pq} \sigma_p \sigma_q, \tag{6}$$

where the sum is restricted to be over nearest-neighbor pairs $pq$ of spins.

It is easy to see the model exhibits some of the qualitative characteristics of real magnetic systems. There are two lowest-energy states which we can call $+$ and $-$. In these states all spins are either up ($+$ state) or down ($-$ state). In either case the energy of the state is

$E_0 = -J \times (\#$ of nearest-neighbor bonds$)$

$\quad = -J \times (\#$ of atoms$) \times (\#$ of bonds per spin$).$ (7)

In three dimensions, for example, each spin has 6 nearest neighbors but one must divide by 2 to avoid counting bonds twice. So

$$E_0 = -3NJ, \tag{8}$$

where $N$ is the number of spins. In both of these states the spins are all aligned and so there is a large magnetization.

Since at zero temperature there is a probability of 0.5 of finding the system in either of these two lowest-energy states we can see that the Ising model does indeed exhibit a spontaneous magnetization. Higher-energy states, which will be important at nonzero temperatures, are produced by reversing the direction of some of the spins. It is clear that these states will have a smaller spontaneous magnetization, and so we expect that the magnitude of the spontaneous magnetization will decrease as the temperature goes up, again in qualitative agreement with what happens for a real magnetic system.

To "solve" the Ising model means to evaluate the sum for $Z$ [Eq. (1)] using the expression for the energy levels [Eq. (6)]. This can be done with not too much difficulty for the 1D model with the result[4]

$$Z = [2 \cosh(J/k_BT)]^N. \tag{9}$$

For the 2D model the problem is much harder, and the solution was first given in a celebrated paper by Onsager.[5] Although his method has been simplified somewhat[6] there is still no easy derivation of the result. He found that

$$Z = [2 \cosh(2J/k_BT)e^I]^N, \tag{10}$$

where

$$I = (2\pi)^{-1} \int_0^\pi d\phi \ln\{(1/2)[1 + (1 - \kappa^2 \sin^2\phi)^{1/2}]\}, \tag{11}$$

$$\kappa = 2 \sinh(2J/k_BT)/\cosh^2(2J/k_BT). \tag{12}$$

In 3D no analytic solution has yet been found, and there are only numerical calculations of the thermodynamic quantities. From $Z$ one can calculate the internal energy $U$ and the specific heat $C$ using the formulas

$$U = k_BT^2 \, d \ln Z/dT, \tag{13}$$

$$C = dU/dT. \tag{14}$$

For the 1D model one finds there is no phase transition in the sense that $U$ and $C$ are both smoothly varying functions of $T$. The increase in the alignment of the spins as the temperature decreases occurs in a very gradual way. In 2D the specific heat is as shown in Fig. 2. There is a singularity at the temperature $T_c$ such that
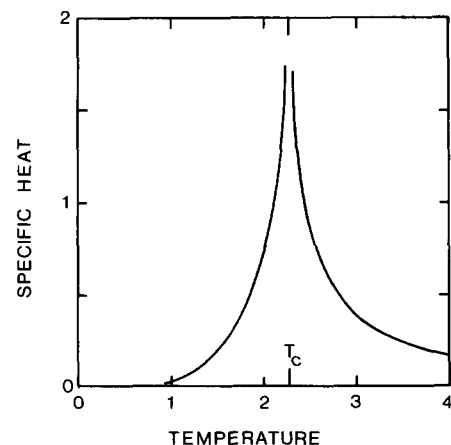


Fig. 2. Specific heat of the 2D Ising model as a function of temperature. The specific heat is plotted in units of $Nk_B$, and the temperature in units of $J/k_B$. In these units the critical temperature is 2.269.
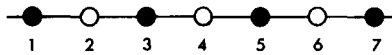
Fig. 3. Section of a 1D Ising model. The sums over the spin variables $\sigma_2$, $\sigma_4$, $\sigma_6$, etc. (spins denoted by open circles) are performed first. This leaves a new sum that involves only the spin variables $\sigma_1$, $\sigma_3$, $\sigma_5$, etc. (solid circles).

$$\sinh(2J/k_B T_c) = 1. \qquad (15)$$

Near this temperature $C$ has a singular contribution which varies with temperature as

$$(8k_B/\pi)(J/k_B T_c)^2 \ln|1/(T-T_c)| \qquad (16)$$

In 3D there is also a phase transition and the singular term in the specific heat is proportional to

$$|T - T_c|^{-\alpha}, \qquad (17)$$

where $\alpha$ is close to 0.125.

## RENORMALIZATION GROUP

The aim of the renormalization group theory is to calculate directly the critical exponents such as $\alpha$ and $\beta$ occurring in Eqs. (2), (3), (4), and (17). If the theory also happens to give the partition function at all temperatures so much the better, but the primary emphasis is on the determination of the partition function in the critical region, i.e., for $T$ close to $T_c$. The renormalization group arose from Wilson's brilliant translation[7] of the conceptual picture of phase transitions which arose in the 1960s into a calculational tool. The history of these developments is complex and involves the appreciation of such concepts as scaling, universality, and correlations in the critical region.[8] The approach we want to present here is deliberately directed away from these concepts. In this way we certainly lose both history and what many current workers in the field consider to be the deep physics of the renormalization group (RG). What we gain is the possibility of explaining the RG to a wider audience. We will simply take the view that the RG approach is a practical but approximate way of calculating the sums involved in $Z$.

Consider first the 1D Ising model.[9] We show a section of the chain of spins in Fig. 3. The problem is to evaluate the sum

$$Z = \sum \exp[J(\cdots \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_5 + \cdots)/k_B T]. \qquad (18)$$

The sum is over all possible values of $\sigma_1$, $\sigma_2$, etc. For convenience let us introduce a quantity $K$, called the coupling constant, defined by

$$K \equiv J/k_B T. \qquad (19)$$

We partition $Z$ into the form

$$Z = \sum \cdots e^{K(\sigma_1\sigma_2+\sigma_2\sigma_3)} e^{K(\sigma_3\sigma_4+\sigma_4\sigma_5)} \cdots. \qquad (20)$$

The only place $\sigma_2$ appears in this equation is in the first exponential. We therefore carry out the sum over $\sigma_2$ with the result

$$Z = \sum \cdots [e^{K(\sigma_1 + \sigma_3)} + e^{-K(\sigma_1+\sigma_3)}] \times e^{K(\sigma_3\sigma_4+\sigma_4\sigma_5)} \cdots. \qquad (21)$$

We continue by summing over $\sigma_4$, and $\sigma_6$, etc. This gives

$$Z = \sum \cdots [e^{K(\sigma_1+\sigma_3)} + e^{-K(\sigma_1+\sigma_3)}] \times [e^{K(\sigma_3+\sigma_5)} + e^{-K(\sigma_3+\sigma_5)}] \cdots. \qquad (22)$$

The sum is now over the possible values of the remaining spin variables, i.e., $\sigma_1$, $\sigma_3$, $\sigma_5$, etc. The next idea is to try to find a value of $K'$ and a function $f$ such that

$$e^{K(\sigma_1+\sigma_3)} + e^{-K(\sigma_1+\sigma_3)} \equiv f(K)e^{K'\sigma_1\sigma_3} \qquad (23)$$

for all possible values of $\sigma_1$ and $\sigma_3$. The function $f$ must not depend on $\sigma_1$ or $\sigma_3$. The solution is (see Appendix)

$$K' = (\tfrac{1}{2})\ln\cosh(2K), \qquad (24)$$

$$f(K) = 2\cosh^{1/2}(2K). \qquad (25)$$

Thus we can write

$$Z = \sum \cdots f(K)e^{K'\sigma_1\sigma_3} f(K)e^{K'\sigma_3\sigma_5} \cdots$$
$$= f(K)^{N/2} \sum e^{K'(\cdots \sigma_1\sigma_3+\sigma_3\sigma_5+\cdots)}. \qquad (26)$$

We still have not done the sum, but we notice that if we had been trying to solve a problem where the coupling constant had the value $K'$ and there were only $N/2$ spins exactly this sum would appear. Thus we have shown that the partition function $Z(N,K)$ for $N$ spins interacting with coupling constant $K$ is related to the partition function $Z(N/2,K')$ for $N/2$ spins and coupling constant $K'$ by the equation

$$Z(N,K) = f(K)^{N/2}Z(N/2,K'). \qquad (27)$$

For a large system we know how $Z$ depends on $N$. Since we believe that the free energy $F$ is proportional to the size of the system[10] it must be true that

$$\ln Z = N\zeta, \qquad (28)$$

where $\zeta$ depends on $K$ but is independent of the system size. Then from Eq. (27)

$$\zeta(K) = (1/2)\ln f(K) + (1/2)\zeta(K') \qquad (29)$$

$$\therefore \ \zeta(K') = 2\zeta(K) - \ln[2\cosh^{1/2}(2K)]. \qquad (30)$$

Equations (24) and (30) are the essential results of the renormalization group[11] analysis. If the partition function is known for one value of the coupling constant $K$, or equivalently the temperature $T$, these equations provide a recursion relation that can be used to calculate $Z$ for other values. Using Eqs. (24) and (30) this recursion, or "renormalization," is always towards lower values of $K$, i.e., $K'$ is always less than $K$. One can find recursion relations that work in the opposite direction by solving Eq. (24) for $K$. These recursion relations are

$$K = (1/2)\cosh^{-1}(e^{2K'}), \qquad (31)$$

$$\zeta(K) = (1/2)\ln 2 + (1/2)K' + (1/2)\zeta(K'). \qquad (32)$$

These results can be used to find the partition function $\zeta(K)$ in the following way. For $K'$ very small, e.g., 0.01, the interaction between the spins is negligible. For free spins the partition function is just the number of ways of arranging them. So for $K' = 0.01$

$$Z \approx 2^N \qquad (33)$$

$$\therefore \ \zeta(0.01) \approx \ln 2. \qquad (34)$$

We now calculate $K$ from Eq. (31) and obtain the result 0.100 334. The value of $\zeta$ for this value of $K$ is found from Eq. (32) to be 0.698 147. The procedure is then repeated with $K'$ equal to 0.100 334, and by continuing in this way one obtains the results shown in Table I. This table includes

Table I. Values of $\zeta$ for the 1D Ising model calculated from the recursion formulas Eqs. (31) and (32) of the renormalization group and from the exact formula derived from Eq. (9). $\zeta$ is related to the partition function $Z$ through Eq. (28).

| $K$ | $\zeta(K)$ Renormalization group | $\zeta(K)$ Exact |
|---|---|---|
| 0.01 | ln 2 | 0.693 197 |
| 0.100 334 | 0.698 147 | 0.698 172 |
| 0.327 447 | 0.745 814 | 0.745 827 |
| 0.636 247 | 0.883 204 | 0.883 210 |
| 0.972 710 | 1.106 299 | 1.106 302 |
| 1.316 710 | 1.386 078 | 1.386 080 |
| 1.662 637 | 1.697 968 | 1.697 968 |
| 2.009 049 | 2.026 876 | 2.026 877 |
| 2.355 582 | 2.364 536 | 2.364 537 |
| 2.702 146 | 2.706 633 | 2.706 634 |

the exact values of $\zeta$ calculated from Ising's formula Eq. (9). The agreement is excellent. A remarkable feature of this method is that small errors in the first value of $\zeta$ actually lead to smaller and smaller errors as the calculation proceeds. If one attempts the same calculation starting from a large coupling constant, on the other hand, one obtains progressively larger errors.

This process may be represented by a "flow diagram," as shown in Fig. 4, which shows how $K$ moves under successive recursion. Of significance are points where recursion does not change $K$. For the 1D Ising model the only such "fixed points" are at $K = 0$ and $K = \infty$.

To use the renormalization group to study phase transitions we must apply it to the 2D Ising model, as shown in Fig. 5. In analogy with the 1D problem the first step is to sum over the spin variables of half of the spins. A choice of which spins to sum over is shown in Fig. 5. After the summation the expression for $Z$ becomes

$$Z = \sum \cdots [e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}] \cdots . \quad (35)$$

The sum is over all possible values of the remaining spin variables. There is a square bracket term for every large square, such as 1234, in the lattice of remaining spins. To follow the same method as in the 1D problem we would now like to do something so that $Z$ involves exactly the same form of difficult summation that was present in the original expression. This is not possible, however. In fact, it is straightforward to show that Eq. (35) is equivalent to the result (see Appendix)

$$Z = f(K)^{N/2} \sum \exp\!\Big(K_1 \sum_{nn} \sigma_p\sigma_q + K_2 \sum_{nnn} \sigma_p\sigma_q + K_3 \sum_{sq} \sigma_p\sigma_q\sigma_r\sigma_s\Big). \quad (36)$$

where
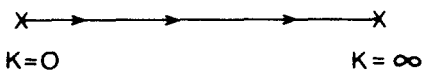
$$f(K) = 2 \cosh^{1/2}(2K) \cosh^{1/8}(4K), \quad (37)$$



Fig. 4. Flow diagram for the 1D Ising model. The only fixed points are at $K = 0$ and $\infty$ and are marked ✕. The arrows show the direction of flow when the recursion formula Eq. (31) is used. Equation (24) gives a flow in the opposite direction.
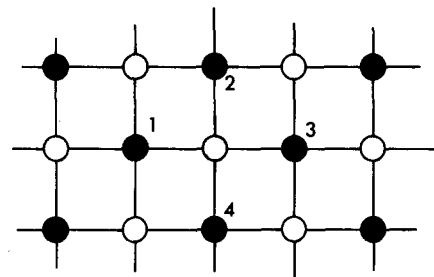


Fig. 5. Section of the 2D Ising model. The summed over spins are denoted by open circles, and the spins that remain by solid circles. Notice that the lattice formed by the remaining spins is still a simple cubic lattice, but is rotated by 45° relative to the original lattice.

and the sums in the exponential are over all nearest-neighbor pairs, next-nearest-neighbor pairs, and sets of four spins $pqrs$ around all squares. Explicit expressions for $K_1$, $K_2$, and $K_3$ are

$$K_1 = (1/4)\ln \cosh(4K), \quad (38)$$

$$K_2 = (1/8)\ln \cosh(4K), \quad (39)$$

$$K_3 = (1/8)\ln \cosh(4K) - (1/2)\ln \cosh(2K). \quad (40)$$

At this point an essential difference between the renormalization group in the 1D and 2D models appears. The expression for $Z$ in the 2D case [Eq. (36)] does not just involve the same sort of sum that appeared in the original expression for $Z$. Hence, unless we can somehow get rid of $K_2$ and $K_3$ the method fails. To eliminate $K_2$ and $K_3$ it is necessary to make some sort of approximation, and this requires considerable intuition. For example, two possibilities are the following.

(1) Simply ignore $K_2$ and $K_3$. This gives recursion relations

$$K' = (1/4)\ln \cosh(4K), \quad (41)$$

$$\zeta(K') = 2\zeta(K) - \ln[2 \cosh^{1/2}(2K) \cosh^{1/8}(4K)]. \quad (42)$$

These recursion relations have the same sort of flow pattern as in the 1D model (Fig. 4). They do not predict a phase transition, i.e., $\zeta$ turns out to be an analytic function of $K$.

(2) A better result is obtained by correcting the theory in an approximate way for the presence of the term in $K_2$. $K_2$ is the coupling between next-nearest spins, such as 1 and 3. Both $K_1$ and $K_2$ are positive, and hence both have the effect of increasing the alignment of spins. Hence, a possible procedure is to drop $K_2$ but simultaneously increase $K_1$ to a new value $K'$ so that the net "aligning tendency" remains the same. A crude way of determining $K'$ is as follows. Consider the energy of the lattice when all the spins are aligned. In a lattice of $N/2$ spins there are $N$ nearest-neighbor bonds and $N$ next-nearest bonds. Thus if we retain $K_1$ and $K_2$ the total energy is

$$-Nk_BTK_1 - Nk_BTK_2.$$

We choose $K'$ so that this gives by itself the same energy. Thus

$$K' = K_1 + K_2$$

$$= (3/8)\ln \cosh(4K). \quad (43)$$

The partition function is still determined by Eq. (42). The recursion relations now have a remarkable new feature. The
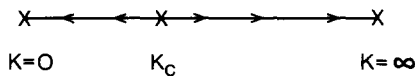
Fig. 6. Flow diagram for the 2D Ising model based upon the recursion formula (43).

flow diagram, as shown in Fig. 6, splits into two separate parts separated by a fixed point at $K_c = 0.506\ 98$. If one starts with a $K$ just to the right of this point recursion increases $K$ eventually to $\infty$. Starting just to the left decreases $K$ to zero. Detailed study of the partition function using these recursion relations shows that there is a singularity in $\zeta$ and $Z$ at $K_c$, and so this fixed point is to be associated with the phase transition. The value of $K_c$ is surprisingly near the exact value (much nearer than it should be considering the naivete of the approximation). The exact solution of Onsager [Eq. (15)] shows that the correct value of $K_c$ is

$$J/k_B T_c = (1/2)\sinh^{-1}(1)$$
$$= 0.440\ 69. \qquad (44)$$

Another test is to calculate the specific heat. By expanding $\zeta$ and $Z$ around $K_c$ it is a simple exercise (see Appendix) to show that near the transition

$$C \propto |1 - T/T_c|^{-\alpha}, \qquad (45)$$

where

$$\alpha = 2 - \ln 2/\ln (dK'/dK|_{K=K_c}), \qquad (46)$$

$\alpha$ comes out to be 0.131. This is to be compared with Onsager's exact result, which has a logarithmic singularity in the specific heat, and hence a critical exponent $\alpha$ of zero.

To obtain a more accurate value of the critical exponent is not trivial. In fact, the second example given above is deceptive in the sense that seemingly logical improvements can easily lead to worse results! For example, if $K'$ is calculated in the same way as above but the energy contribution from $K_3$ in the aligned state is included a poorer value of $\alpha$ is obtained. For a discussion of better approximation schemes see Ref. 11.

Our experience has been that this material can be covered at a fairly leisurely pace in two lectures, and successfully gives students an idea of what the renormalization group is, and how it can be used in the theory of second-order phase transitions.

## APPENDIX

We give here some of the intermediate steps in the derivations. Equations (24) and (25) are obtained as follows. Equation (23) must be true for all values of $\sigma_1$ and $\sigma_3$. For $\sigma_1 = 1$ and $\sigma_3 = 1$, Eq. (23) becomes

$$e^{2K} + e^{-2K} = fe^{K'}. \qquad (A1)$$

The same result is obtained with $\sigma_1 = -1$ and $\sigma_3 = -1$. For $\sigma_1 = 1$ and $\sigma_3 = -1$ or $\sigma_1 = -1$ and $\sigma_3 = 1$ we obtain

$$2 = fe^{-K'}. \qquad (A2)$$

Solution of Eqs. (A1) and (A2) leads to the expressions for $K'$ and $f$ given in the text.

The derivation of Eqs. (36)–(40) proceeds in a similar way. We first try to find $K_1$, $K_2$, $K_3$, and $f$ such that

$$e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}$$
$$= f\exp[(1/2)K_1(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1)$$
$$+ K_2(\sigma_1\sigma_3 + \sigma_2\sigma_4) + K_3\sigma_1\sigma_2\sigma_3\sigma_4] \qquad (A3)$$

for all possible values of $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. For example, when all the $\sigma$'s are equal to $+1$, or all equal to $-1$, we obtain the condition

$$e^{4K} + e^{-4K} = fe^{2K_1+2K_2+K_3}. \qquad (A4)$$

Investigation of all of the other possible values of the $\sigma$'s gives the conditions

$$2 = fe^{-2K_1+2K_2+K_3}, \qquad (A5)$$
$$e^{2K} + e^{-2K} = fe^{-K_3}, \qquad (A6)$$
$$2 = fe^{-2K_2+K_3}. \qquad (A7)$$

Solution of these equations gives the expressions for $K_1$, $K_2$, $K_3$, and $f$ given in the text [Eqs. (38)–(40)]. The result for $Z$ now becomes

$$Z = \sum \cdots [f\exp((1/2)K_1(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1)$$
$$+ K_2(\sigma_1\sigma_3 + \sigma_2\sigma_4) + K_3(\sigma_1\sigma_2\sigma_3\sigma_4))] \cdots . \qquad (A8)$$

There is one square bracket for each large square of the type 1234. Since there are $N/2$ such squares $Z$ contains a factor

$$f^{N/2}.$$

A nearest-neighbor term such as

$$e^{(1/2)K_1\sigma_1\sigma_2}$$

will appear both in the square bracket that has been written out explicitly and in one other square bracket. Thus the *total* term coming from the neighbors 1 and 2 is

$$e^{K_1\sigma_1\sigma_2}.$$

Hence there appears in $Z$ a factor

$$\exp\left(K_1 \sum_{nn} \sigma_p\sigma_q\right).$$

The understanding of the terms in $K_2$ and $K_3$ that appear in Eq. (36) is straightforward.

To derive Eq. (46) assume that there is a nonanalytic term in $\zeta(K)$ which is

$$a|K - K_c|^{2-\alpha}$$

where $a$ is a constant. Near to $K_c$ we have to first order in $(K - K_c)$

$$K' = K_c + (K - K_c)\frac{dK'}{dK}\bigg|_{K=K_c}. \qquad (A9)$$

Now expand Eq. (42) around $K = K_c$. On the right-hand side the only term which is a $(2 - \alpha)$ power is

$$2a|K - K_c|^{2-\alpha}. \qquad (A10)$$

On the left-hand side the corresponding term is

$$a|K' - K_c|^{2-\alpha} = a\left|(K - K_c)\frac{dK'}{dK}\bigg|_{K=K_c}\right|^{2-\alpha} \qquad (A11)$$

using Eq. (A9). Comparison of Eqs. (A10) and (A11) gives

$$2a|K - K_c|^{2-\alpha} = a|K - K_c|^{2-\alpha} \left.\frac{dK'}{dK}\right|_{K=K_c}^{2-\alpha}, \quad \text{(A12)}$$

and Eq. (46) follows.

[1]F. Reif, *Statistical and Thermal Physics* (McGraw-Hill, N.Y., 1965).

[2]See, for example, Chap. 8 of Ref. 1.

[3]Choice of the $x$-$y$ plane is completely arbitrary since there is no interaction between the spin direction and the plane in which the spins are arranged.

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[8]Some review papers from this period are the following: P. Heller, Rep. Prog. Phys. **30**, 731(1967); M. Fisher, *ibid.* p. 615; L. Kadanoff *et al.*, Rev.Mod. Phys. **39**, 395 (1967); L. Kadanoff, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green, (Academic, New York, 1976), Vol. 5A; S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).

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[10]An explicit proof of this point for some spin systems has been given by R. B. Griffiths, J. Math. Phys. **5**, 1215 (1964).

[11]This particular type of application is called the "real-space" renormalization method. For a review see Th. Niemeijer and J. M. J. Van Leeuwen in Vol. 6 of the Domb-Green series mentioned in Ref. 8.