Last time:

{Lecture 10}

Finite interval (1D): $0 \quad x_0 \quad L$

1. Survival prob. $S(t) \underset{t \text{ large}}{\sim} e^{-\frac{D\pi^2 t}{L^2}} \equiv e^{-t/\tau}$,

$$\tau = \frac{L^2}{D\pi^2} \sim \frac{L^2}{D}$$

diffusion time-scale

2. Eventual exit prob. ('Laplace domain)

$$\underset{\text{at } x=0}{E_-^{(x_0)}} = 1 - \frac{x_0}{L}, \qquad \underset{\text{at } x=L}{E_+^{(x_0)}} = \frac{x_0}{L}$$

$$E_+ + E_- = 1 \quad (\text{"everybody dies"})$$

3. Same result can be obtained using path decomposition:

$E_+(x_0) = \sum_{\text{paths}} \Pi(x_0) = \frac{1}{2}\sum_{\text{paths}'} \Pi(x+dx) + \frac{1}{2}\sum_{\text{paths}''} \Pi(x-dx)$,

$E''(x_0) = 0$ [unbiased RW]

or $E_+(x_0) = \frac{1}{2}E_+(x_0+dx) + \frac{1}{2}E_+(x_0-dx)$

gives E_+, E_- w/ appropr. BCs

same for E_- ...

$dx \; dx$

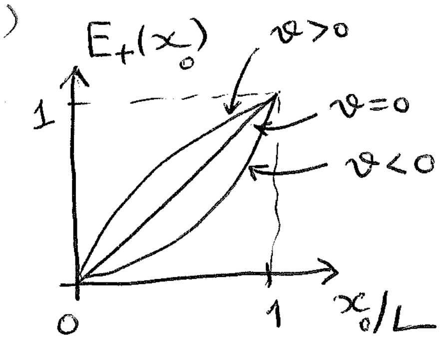
$0 \quad x_0 \quad L$

jump probs. $\begin{cases} \frac{1}{2} \quad \frac{1}{2} & \text{unbiased RW} \\ q \quad p & \text{biased RW} \end{cases}$

Biased RW: $E(x) = pE(x+dx) + qE(x-dx)$

$$\underbrace{D}_{\frac{(dx)^2}{dt}} E'' + \underbrace{v}_{(p-q)\frac{dx}{dt}} E' = 0$$

$$E_{+(x_0)} = \frac{1 - e^{-v x_0/D}}{1 - e^{-\underbrace{vL/D}_{\text{Peclet \#}}}}$$



$E(x) \simeq \overset{=1}{(p+q)} E(x) + (p-q)E'(x)\,dx + \left[\frac{p}{2}E''(x) + \frac{q}{2}E''(x)\right](dx)^2$,

or $DE'' + vE' = 0$, where $\begin{cases} v = (p-q)\frac{dx}{dt}, \\ D = \frac{(dx)^2}{2dt}. \end{cases}$

3. Consider $t(x) \leftarrow \langle \text{time} \rangle$ to reach $\emptyset$ or L starting from x

$$t(x) = \sum_{\text{paths}} \overbrace{\Pi(x)}^{\text{path prob.}} \underbrace{t_p(x)}_{\text{path time}} \ominus$$

$$\ominus \; \frac{1}{2} \sum_{\text{paths}'} \Pi(x+dx)\,[dt + t_p(x+dx)] +$$

$$+ \frac{1}{2} \sum_{\text{paths}''} \Pi(x-dx)\,[dt + t_p(x-dx)] =$$

$$= dt + \frac{1}{2} t(x+dx) + \frac{1}{2} t(x-dx).$$

$$\frac{1}{2} \sum_{\text{paths}'} \Pi(x+dx) + \frac{1}{2} \sum_{\text{paths}''} \Pi(x-dx) =$$

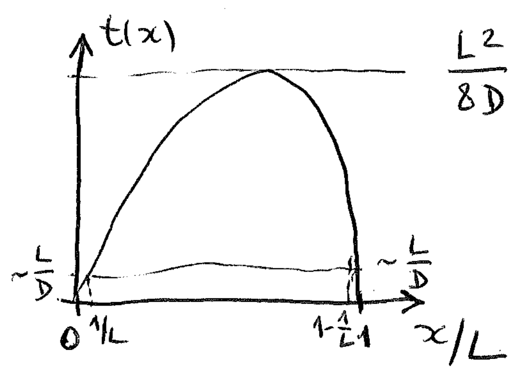
$$= \underbrace{\sum_{\text{paths}}}_{\text{as before}} \Pi(x) = \underbrace{1}_{\text{all paths, to } \emptyset \text{ or } L}$$

So, $\cancel{t(x)} \simeq dt + \cancel{t(x)} + \frac{1}{2} t''(x)(dx)^2$, or

$$t''(x) = -\frac{2dt}{(dx)^2} = -\frac{1}{D}$$

↑ 1st moment eq'n from the "formal" solution

$$t(0) = t(L) = 0 \Rightarrow t(x) = \frac{x(L-x)}{2D}.$$



parabola

Two time scales:

$\sim \frac{L^2}{D}$ far from the boundaries

$\sim \frac{1}{2D} \, 1(L-1) \sim \frac{L}{D} \ll \frac{L^2}{D}$

"memory" or "boundary" effect

(4.) Now consider $t_+(x)$ & $t_-(x)$.

$$t_+(x) = \sum_{\substack{\text{paths}\\ x\to L}} \Pi(x)\, t_p(x) \Big/ \underbrace{\sum_{\substack{\text{paths}\\ x\to L}} \Pi(x)}_{E_+(x)}$$

So, $$E_+(x)t_+(x) = \frac{1}{2}\sum_{\substack{\text{paths}\\ x+dx\to L}} \Pi(x+dx)\,[dt + t_p(x+dx)] +$$

$$+ \frac{1}{2}\sum_{\substack{\text{paths}\\ x-dx\to L}} \Pi(x-dx)[dt + t_p(x-dx)] =$$

$$= dt\, E_+(x) + \frac{1}{2} t_+(x+dx)\, E_+(x+dx) +$$
$$+ \frac{1}{2} t_+(x-dx)\, E_+(x-dx) \,.$$

Expand as before:

$$\cancel{E_+ t_+} \simeq dt\, E_+ + \cancel{E_+ t_+} + \frac{1}{2}(t_+E_+)''(dx)^2,$$

$$D\,(t_+E_+)'' = -E_+$$

$$\text{BCs} \Rightarrow \begin{cases} E_+(L)\,\overbrace{t_+(L)}^{=0} = 0, \\ \underbrace{E_+(0)}_{=0}\, t_+(0) = 0 \end{cases}$$

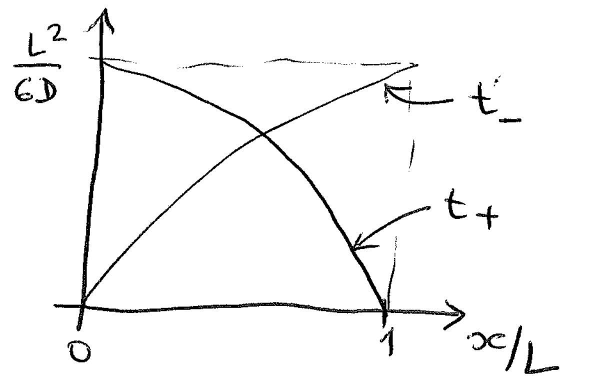
Solution: $$t_+(x) = \frac{L^2 - x^2}{6D}.$$

Indeed, $$E_+t_+ = \frac{x}{L}\,\frac{L^2-x^2}{6D}$$

$$(E_+t_+)' = \frac{1}{L}\,\frac{L^2-x^2}{6D} - 2x\,\frac{x}{L}\,\frac{1}{6D} =$$
$$= \frac{1}{L}\,\frac{L^2-x^2}{6D} - \frac{x^2}{3DL} = \text{const} - \frac{x^2}{2DL}.$$

So, $\left(E_+ t_+\right)'' = -\dfrac{x}{DL} = -\dfrac{E_+}{D}$ as expected

Similarly, $t_-(x) = \dfrac{L^2 - (L-x)^2}{6D}$



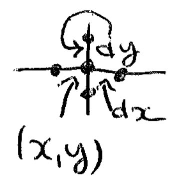
Note that

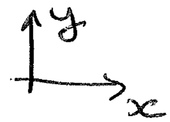
$$t(x) = E_+(x)\,t_+(x) + E_-(x)\,t_-(x) =$$
$$= \frac{x}{L}\,\frac{L^2 - x^2}{6D} + \left(1 - \frac{x}{L}\right)\frac{L^2 - (L-x)^2}{6D} =$$
$$= \frac{xL^2 - x^3 + (L-x)(2Lx - x^2)}{6DL} =$$
$$= \frac{xL^2 - \cancel{x^3} + 2L^2x - Lx^2 - 2Lx^2 + \cancel{x^3}}{6DL} =$$
$$= \frac{Lx - x^2}{2D} =$$
$$= \frac{x(L-x)}{2D},$$

as expected

Now consider multiD problems:

Consider 2D





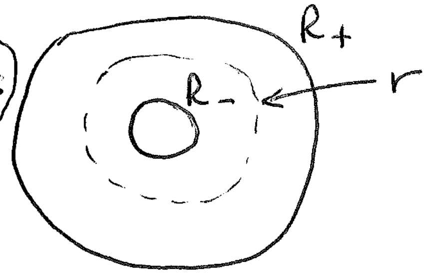
$$E(x,y) = \tfrac{1}{4}E(x, y+dy) + \tfrac{1}{4}E(x, y-dy) + \tfrac{1}{4}E(x+dx, y) + \tfrac{1}{4}E(x-dx, y)$$

gives $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)E(x,y) = 0.$

splitting prob. to each of the 4 boundaries

Spherical geometry:

d dimensions



Start at r (spher. symm. initial cond'n), diffuse until R_- or R_+ is hit

So, consider $\nabla^2 E_{\pm}(r) = 0$

$$\text{BCs}: \begin{cases} E_-(R_-) = 1, & E_-(R_+) = 0 \\ E_+(R_-) = 0, & E_+(R_+) = 1 \end{cases}$$

Spher. symm.: $\frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left(r^{d-1} \frac{\partial}{\partial r}\right) E(r) = 0$

Solved by
$$\begin{cases} E(r) = A + \frac{B}{r^{d-2}}, & d \neq 2 \\ E(r) = A + B \log r, & d = 2 \end{cases}$$

E.g. $d=1$ $\quad E(r) = A + Br$

$$\frac{\partial^2}{\partial r^2} E(r) = 0 \quad .$$

$d=2$ $\quad \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) \log r = 0 \quad , \ldots$

Invoke BCs:
$$E_+(r) = \begin{cases} \dfrac{1 - (R_-/r)^{d-2}}{1 - (R_-/R_+)^{d-2}}, & d \neq 2 \\[2ex] \dfrac{\log(r/R_-)}{\log(R_+/R_-)}, & d = 2 \end{cases} \qquad (*)$$

$$E_-(r) = 1 - E_+(r)$$

$R_+ \to \infty$ limit:

$d>2$: $\quad E_-(r) = \left(\frac{R_-}{r}\right)^{d-2} \qquad \left(\text{e.g. } \sim \frac{1}{r} \text{ if } d=3\right)$

$R_- \to 0$ limit:

$d>2$: $\quad E_+(r) \to 1$

"Affinity" for the boundary:

Set $E_+(r_{eq}) = E_-(r_{eq}) = \frac{1}{2}$ in (*) $\Leftrightarrow$ $\dfrac{1 - \frac{1+(R_-^{d-2}/R_+^{d-2})}{2}}{1-(R_-^{d-2}/R_+^{d-2})} = \frac{1}{2}$, as expected.

$$r_{eq} = \begin{cases} \left(\frac{1}{2}\right)^{\frac{1}{d-2}} \cancel{}\left(R_-^{2-d} + R_+^{2-d}\right)^{1/(2-d)}, & d \neq 2 \\ \sqrt{R_- R_+}, & d = 2 \end{cases}$$

$\leftarrow$ geom. mean of R_+ & R_-

$$\frac{1 - \frac{R_-^{d-2}}{2\left(R_-^{2-d}+R_+^{2-d}\right)^{-1}}}{1 - \frac{R_-^{d-2}}{R_+^{d-2}}} = \Leftrightarrow$$

Indeed, $\dfrac{\log\left(\sqrt{R_- R_+}/R_-\right)}{\log(R_+/R_-)} = \dfrac{1}{2}$

R_- finite, $R_+ \to \infty$:

$$r_{eq} \to \begin{cases} R_- \times 2^{1/(d-2)} & d > 2 \\ R_+ \times 2^{1/(d-2)} & d < 2 \\ \sqrt{R_- R_+} & d = 2 \end{cases}$$

<u>$d>2$</u>: $r_{eq} \simeq R_-$ (e.g. $d=3 \Rightarrow r_{eq} \to 2R_-$), the particle will wander to ∞ unless it starts close to the inner sphere.

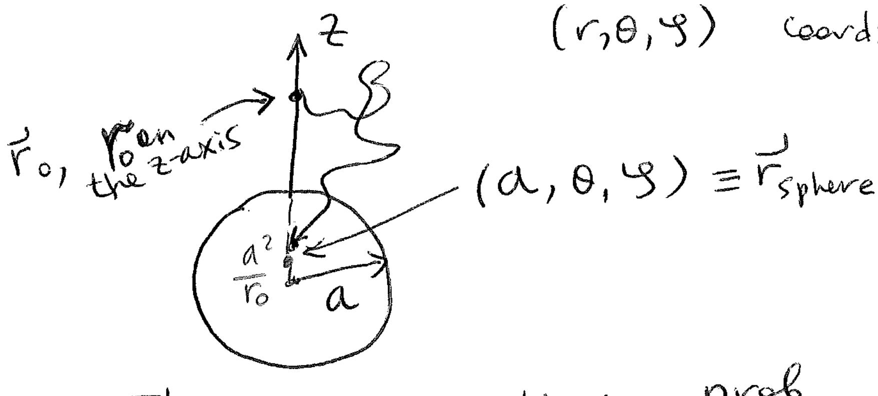
<u>$d<2$</u>: $r_{eq} \simeq R_+ 2^{\frac{1}{d-2}}$, e.g. $d=1$:

$$r_{eq} \simeq R_+ \frac{1}{2}.$$

So, d acts as effective "bias".

Now consider <u>where</u> on the sphere the particle hits.

(r, θ, φ) coords



The corresponding prob. is the <u>same</u> as electric field at $\vec{r}$ when a point charge of magnitude $q = \frac{1}{\Omega_d D}$ is placed at $\vec{r}_0$ and the surface of the sphere is grounded (i.e. at $\emptyset$ potential).

Use image method:
$(d > 2)$

$$\varphi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}_0|^{d-2}} + \frac{q'}{|\vec{r} - \vec{r}_0'|^{d-2}}$$

$q', \vec{r}_0'$ describe the image charge

$$\varphi(\vec{r}_{sphere}) = \emptyset \implies \begin{cases} q' = -q \left(\frac{a}{r_0}\right)^{d-2}, \\ r_0' = \frac{a^2}{r_0} \end{cases}$$

$$\underbrace{\varphi(\vec{r})}_{\varphi(r,\theta)} = q \left[\frac{1}{(r^2 + r_0^2 - 2 r r_0 \cos\theta)^{\frac{d-2}{2}}} - \frac{(a/r_0)^{d-2}}{\left(\frac{a^4}{r_0^2} + r^2 - 2 \frac{a^2 r}{r_0} \cos\theta\right)^{\frac{d-2}{2}}} \right]$$

$\varphi(\vec{r}_{sphere}) = 0$ as expected.

Compute radial component of the electric field: $\left(q = \frac{1}{\Omega_d D} \right)$

$D \frac{\partial \varphi(r,\theta)}{\partial r}\Big|_{r=a}$ $(d>2)$

radial component of E-field on the sphere's surface

$$\parallel$$

$$E(\theta) = \frac{d-2}{\Omega_d} \frac{1}{a r_0^{d-2}} \times \frac{1 - \frac{a^2}{r_0^2}}{\left(1 - \frac{2a}{r_0}\cos\theta + \frac{a^2}{r_0^2}\right)^{d/2}}$$

$\int \underbrace{d\Omega}_{2\pi \sin\theta d\theta} E(\theta) = \left(\frac{a}{r}\right)^{d-2}$, as before

$$\frac{E(0)}{E(\pi)} = \frac{\left(1 + \frac{2a}{r_0} + \frac{a^2}{r_0^2}\right)^{d/2}}{\left(1 - \frac{2a}{r_0} + \frac{a^2}{r_0^2}\right)^{d/2}} = \frac{\left(1 + \frac{a}{r_0}\right)^d}{\left(1 - \frac{a}{r_0}\right)^d}.$$

So, if $d = 3$ & $r_0 = 2a$:

$$\frac{E(0)}{E(\pi)} = \frac{(3/2)^3}{(1/2)^3} = 27.$$

The particle is 27 times more likely to hit the North pole rather than the South pole.