Consider random var. \( X(t) \) (scalar or vector); define \( p(x,t|y,0) \) as before.

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {\( V \)};
  \node at (1,0) {\( \bar{V} \)};
  \node at (0.5,0.5) {\( x \)};
  \node at (0.5,0.75) {\( \bar{x} \)};
  \draw[-] (0,0) -- (1,0);
  \draw[-] (0,0) arc (180:0:0.5);
  \draw[-] (1,0) arc (180:0:0.5);
  \draw[-] (0,0) -- (0,0.5);
  \draw[-] (1,0) -- (1,0.5);
  \draw[-] (0,0) -- (1,0);
  \draw[-] (0,0) -- (1,0);
  \node at (0.25,1) {\( \text{Starting point } y \)};
\end{tikzpicture}
\end{center}

1st passage time: time needed to reach \( \bar{V} \) for the 1st time. For ex.: a chemical reaction which needs energy \( E_c \) to occur:

\[ V = \{ E < E_c \}, \quad \bar{V} = \{ E > E_c \} \]

Prob. that \( X(t) \) is still in \( V \) at time \( t \) is given by \( p_{V}(y,t) = \int_{V} dx \, p(x,t|y,0) \)

Define \( \eta(y,t) \) to be prob. density for FPT:

\[ \text{if } \tilde{t} \text{ is the 1st time } X(t) \text{ reaches } \bar{V}, \]
\[ \text{given } X(0) = y \notin V, \quad \eta(y,t)dt \text{ is the prob. that } t < \tilde{t} < t + dt. \]

Note that if \( X(t) \) is in \( V \) at time \( t \),
\[ @t+dt \] it either stays in \( V \) or makes 1st passage.
So, \( p(y,t) - p(y,t+dt) = \eta(y,t)dt \), or

\[
\eta(y,t) = -\frac{\partial p(y,t)}{\partial t}
\]

FPT moments:

\[
\langle t^n(y) \rangle = \int_0^\infty dt \, t^n \eta(y,t) = \]

\[
= n \int_0^\infty dt \, t^{n-1} p(y,t).
\]

\( n = 1, 2, \ldots \)

Now, 1D case:

Theoretically, could solve the FP eq'n w/ absorbing boundary conditions; however, it's easier to define \( \psi(x,t) \) - the prob. that FPT is \( \leq t \), given \( X(0) = x \).

Define \( p(y,dt|x_0)dy \) - the prob. that \( y \leq X(dt) \leq y + dy \), given that \( X(0) = x \).

Then \( \psi(x,t+dt) = \int_0^A dy \, p(y,dt|x_0) \psi(y,t) \)

for \( t \in (0,\infty) \), this transition does not lead to 1st passage.
Expand $y(y,t)$ around $x$:

$$y(y,t) = y(x,t) + \frac{\partial y(x,t)}{\partial x} + \frac{1}{2} \frac{\partial^2 y(x,t)}{\partial x^2}$$

Then

$$y(x,t+dt) = y(x,t) \int_{0}^{A} dy \ p(y,dt|x,0) +$$

$$\quad + \frac{\partial y(x,t)}{\partial x} \int_{0}^{A} dy \ (y-x) \ p(y,dt|x,0) +$$

$$\quad + \frac{1}{2} \frac{\partial^2 y(x,t)}{\partial x^2} \int_{0}^{A} dy \ (y-x)^2 \ p(y,dt|x,0) + \ldots ,$$

$$\lim_{dt \to 0} \frac{y(x,t+dt)-y(x,t)}{dt} = \frac{\partial y(x,t)}{\partial t} =$$

$$= A(x) \frac{\partial y(x,t)}{\partial x} + B(x) \frac{\partial^2 y(x,t)}{\partial x^2} \quad (*)$$

$$\{ A(x) = \lim_{dt \to 0} \frac{1}{dt} \int_{0}^{A} dy \ (y-x) \ p(y,dt|x,0) ;$$

$$B(x) = \lim_{dt \to 0} \frac{1}{dt} \int_{0}^{A} dy \ (y-x)^2 \ p(y,dt|x,0) \},$$

Not dependence in $A(x)$ & $B(x)$ & other "backward" eq'n in 1D.

The process is homogeneous.

BCs: $y(0,t) = 1 \quad y(A,t) = 1$. Absorbing ends are.

Compute moments of $y(x,t)$ easier than solving $(*)$ directly.

Define $\mu_j(x) = \int_{0}^{\infty} dt \ t^j \frac{\partial y(x,t)}{\partial t}$

$$y(x,t+dt) - y(x,t) = \tilde{\eta}(x,t) \ dt \quad \text{or}$$

$$\tilde{\eta}(x,t) = \frac{\partial y(x,t)}{\partial t}$$
Now, apply \( \int_0^\infty dt \, t \, \frac{\partial}{\partial t} \left( \frac{\partial y}{\partial t} \right) \) to (*):

\[
\int_0^1 dt \, \frac{\partial}{\partial t} \left( \frac{\partial y}{\partial t} \right) = \frac{\partial y}{\partial t} \bigg|_0^\infty - \int_0^\infty \frac{\partial y}{\partial t} = -y(x, +\infty) + \frac{y(x, 0)}{x} = -1
\]

\[\mu_j(0) = \mu_j(A) = 0 \quad [\text{absorbing}]\]

So, \( A(x) \frac{d\mu_1(x)}{dx} + \frac{B(x)}{2} \frac{d^2 \mu_1(x)}{dx^2} = -1 \),

Likewise, \( j = 2 \):

\[
\int_0^\infty dt \, t^2 \, \frac{\partial}{\partial t} \left( \frac{\partial y}{\partial t} \right) = \frac{\partial y}{\partial t} (x^2) \bigg|_0^\infty - \int_0^\infty \frac{\partial y}{\partial t} = -2 \mu_1(x).
\]

So, \( A(x) \frac{d\mu_2}{dx} + \frac{B(x)}{2} \frac{d^2 \mu_2}{dx^2} = -2 \mu_1 \).

In general,

\[
A(x) \frac{d\mu_j}{dx} + \frac{B(x)}{2} \frac{d^2 \mu_j}{dx^2} = -j \mu_{j-1}.
\]

\[j = 2, 3, \ldots \]

BCs: \( \mu_j(0) = \mu_j(A) = 0 \).

These are ordinary differential eq's, can be solved! (?)

For example, for \( \mu_1(x) \):

\[\mu_1(x)\]
\[
M_1(x) = 2 \frac{\int_0^A e^{-u(y)} dy \int_0^y dz \left[ \frac{e^{u(z)}}{b(z)} \right]}{\int_0^A e^{-u(y)} dy} \frac{\int_0^x e^{-u(y)} dy}{b(z)} - 2 \frac{\int_0^A e^{-u(y)} dy \int_0^y dz \left[ \frac{e^{u(z)}}{b(z)} \right]}{b(z)}
\]

\[
M_1(0) = 0 \quad (\text{obviously})
\]

\[
M_1(A) = 2 \frac{\int_0^A e^{-u(y)} dy \int_0^y dz \left[ \frac{e^{u(z)}}{b(z)} \right]}{b(z)} - 2 \frac{\int_0^A e^{-u(y)} dy \int_0^y dz \left[ \frac{e^{u(z)}}{b(z)} \right]}{b(z)} = 0 \quad \text{as well.}
\]

Finally,

\[
\frac{dM_1}{dx} = 2 C_1 e^{-u(x)} - 2 e^{-u(x)} \int_0^x dz \frac{e^{u(z)}}{b(z)}
\]

\[
\frac{d^2 M_1}{dx^2} = -C_1 e^{-u(x)} \left[ \frac{2}{b(x)} \frac{A(x)}{b(x)} \right] + 2 e^{-u(x)} \left[ z A(x) \frac{e^{u(z)}}{b(x)} \right] x
\]

\[
x \int_0^x dz \frac{e^{u(z)}}{b(z)} - 2 e^{-u(x)} \frac{e^{u(0)}}{b(x)} =
\]

\[
= 2 \frac{A(x)}{b(x)} e^{-u(x)} \left[ 2 \int_0^x dz \frac{e^{u(z)}}{b(z)} - C_1 \right] - \frac{2}{b(x)}
\]

Now,

\[
\frac{B(x)}{2} \frac{d^2 M_1}{dx^2} + A(x) \frac{dM_1}{dx} = A(x) e^{-u(x)} \left[ 2 \int_0^x dz \frac{e^{u(z)}}{b(z)} \right] - C_1 - 1 \frac{d}{dx} + A(x) C_1 e^{-u(x)} - 2 A(x) x e^{-u(x)} \int_0^x dz \frac{e^{u(z)}}{b(z)} = -1
\]
So, the sol'n is valid.

Finally, consider \( A(x) = 0 \) & \( B(x) = 2D \),\n\( D - \) diff'n const

Then \( \frac{d^2 \mu_1}{dx^2} = -\frac{1}{D} \)

\[ = \] at both 0 & A

With absorbing BC's,
\[ M_1(x) = \frac{1}{2D} x(A-x) \] is the sol'n.

\[ \text{MFPT}(0) \]

Now consider diffusion with drift:
\( A(x) = -\nu x \), \( B = 2D \)

Note: only \( x = 0 \) is absorbing now

From above, \( \mu_1(x) = C_1 \int_0^x dy \frac{e^{-\frac{vy}{D}}}{D} \)

\[ = \frac{1}{D} \int_0^x dy \frac{e^{-\frac{vy}{D}}}{D} \left( \frac{D}{\nu} \right) \left( e^{-\frac{vy}{D}} - 1 \right) \]

\[ = C_1 \frac{D}{\nu} \left( e^{-\frac{vx}{D}} - 1 \right) + \frac{1}{\nu} \int_0^x dy \left[ 1 - e^{-\frac{vy}{D}} \right] = \]

\[ = C_1 \frac{D}{\nu} \left( e^{-\frac{vx}{D}} - 1 \right) + \frac{1}{\nu} \left\{ x - \frac{D}{\nu} \left[ e^{-\frac{vx}{D}} - 1 \right] \right\} \]

\( M_1(0) = 0 \) is satisfied for all \( C_1 \).

Moreover, if \( \frac{vA}{D} \) is large, then

\[ M_1(A) = C_1 \frac{D}{\nu} \frac{vA}{D} + \frac{A}{\nu} \frac{D}{\nu} \frac{vA}{D} \approx \]

\( \approx \)
\[ C_1 \approx \frac{1}{v} \quad \text{ (see below)} \]

Indeed,

\[ M_1(x) = \frac{D}{v^2} [e^{\frac{vx}{D}} - 1] + \frac{x}{v^2} - \frac{D}{v^2} [e^{\frac{vx}{D}} - 1] = \frac{x}{v^2} \quad \text{ indep. of } D \]

\[ C_1 = 2 \frac{\int_0^A \int_0^y e^{\frac{yv}{D}} dy \, dz \, e^{-\frac{vz}{D}}}{\int_0^A dy \, e^{\frac{yv}{D}}} = \]

\[ = \frac{1}{D} \int_0^A dy \, e^{\frac{yv}{D}} (-\frac{D}{v}) [e^{\frac{yv}{D}} - 1] \]

\[ = \frac{1}{D} \int_0^A dy [1 - e^{\frac{yv}{D}}] \]

\[ \approx \frac{1}{v} \]