Lecture 14-15

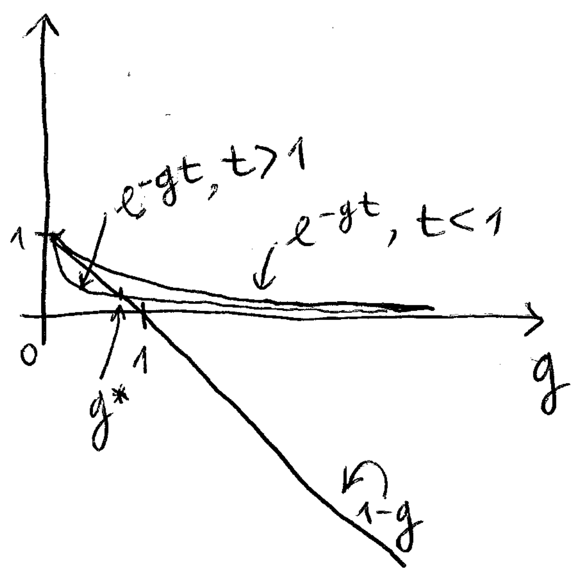
Recall that $M_1 = 1-g \Rightarrow g = 1 - M_1 = 1 - E(y=0,t)$

Since $E e^{-Et} = e^{y-t}$,

$y=0$: $(1-g)\, e^{-(1-g)t} = e^{-t}$, or

$$1-g = e^{-gt} \Rightarrow g = 1 - e^{-gt}.$$

$g=0$ is always a (trivial) solution, while $g \neq 0$ is a non-trivial solution at $t>1$:



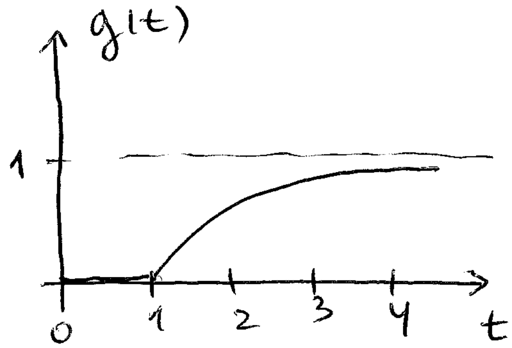
Expansions: $\begin{cases} t \to 1^+ : & t = 1+\delta, \ \delta \text{ small} \\ t \to \infty : & g = 1-\epsilon, \ \epsilon \text{ small} \end{cases}$

Then $g(t) = \begin{cases} 0 & t<1 \\ 2(t-1) - 8(t-1)^2/3 + \dots & t \to 1^+ \\ 1 - e^{-t} - t e^{-2t} + \dots & t \to \infty \end{cases}$

Finally, $\dfrac{dM_0}{dt} = \dfrac{dN}{dt} = \dfrac{g^2-1}{2}$ gives

$$N(t) = \begin{cases} 1 - t/2 , & t \le 1 \\ 1 - t/2 + 2(t-1)^3/3 + \dots , & t \to 1^+ \\ e^{-t} + \frac{t}{2} e^{-2t} + \dots , & t \to \infty \end{cases}$$

$\uparrow$
cluster density
in non-gel phase, remains continuous
across the phase transition



$$\sim \frac{1}{\sqrt{2\pi}\,k^{5/2}} \exp\left[-k(t-1) + k\log t\right] \sim$$

$$\sim \frac{1}{\sqrt{2\pi}\,k^{5/2}} \exp\left[k\varepsilon + k\underbrace{\log(1-\varepsilon)}_{k(-\varepsilon-\frac{\varepsilon^2}{2}\dots)}\right] \sim$$

$t = 1-\varepsilon$
$t-1 = -\varepsilon$

$$\sim \frac{1}{\sqrt{2\pi}} \frac{1}{k^{5/2}} \underbrace{e^{-k\varepsilon^2/2}}_{e^{-k(1-t)^2/2} = e^{-k/k^*}}$$

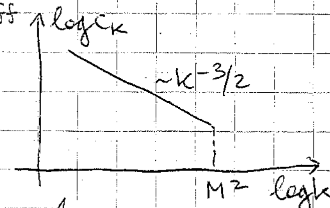
$t<1 \to$ exponential cutoff

Cutoff for ~~…~~ $k^* = \frac{2}{(1-t)^2}$

$t \to 1$: $\boxed{C_k \sim k^{-5/2}}$

$\to$ power-law cutoff — signals gel appearance

Before:



$$C_k \sim \frac{1}{k^{5/2}} \Rightarrow M = \int_1^{k_{max}} k C_k\, dk =$$

$$= \int_1^{k_{max}} \frac{dk}{k^{3/2}} \sim k_{max}^{1/2} \to k_m \sim M^2,$$

"hard cutoff" @ M^2, w/ exponential decay @ around M^2

Adsorption

("flies hit sticky paper & become stuck, no flies on top of flies")

(1D)

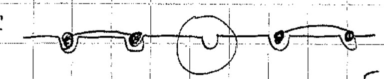
Monomers diffuse & hit the traps

$\dot{p} = 1-p \Rightarrow p(t) = 1 - e^{-t}$ [trivial]

p — fraction of covered sites

Dimer adsorption:

Asymptotic $p(t)$ is <1 b/of frustration:



Expect p_∞ to be between

$2/3$: •—• ○ •—• ○ … What is it? $p_\infty = 1 - e^{-2} \approx 0.865$ ← Flory's result

& 1: •—••—••—• …

Two natural variables

$V_m \equiv$ conc. of vacancies of size $m =$
$= \text{prob}\{\bullet \underbrace{\circ\circ\circ\circ\circ\circ}_{m} \bullet\}$

E_m = conc. of intervals of size $m =$ ← a window of length m, no gain term due to intervals $>m$
$= \text{prob}\{\underbrace{\circ\circ\circ\circ\circ}_{m}\}$ ← prob to have $\geq m$ empty sites — "do not care what's to the left or right"

Eq's: $\dot{V}_m = -(m-1)V_m + 2\sum_{k=2}^{\infty} V_{k+m}$ — non-local eq'n (hard)

$\dot{E}_m = -(m-1)E_m - 2E_{m+1}$, $m \geq 1$ — local eq'n (easy, let's do it) — "drop a dimer in the middle" of an empty line

1st term, RHS: $m-1$ ← # distinct locations where a dimer can absorb

2nd term, RHS: one end of the incident dimer is outside the m-interval

Consider $E_m = e^{-(m-1)t}\varphi(t)$ (ansatz)

Then $\dot{\varphi}\, e^{-(m-1)t} = -2\varphi\, e^{-mt}$,

$\dot{\varphi} = -2\varphi e^{-t}$,

$\frac{d}{dt}(\log\varphi) = -2e^{-t}$,

$\log\left(\frac{\varphi(t)}{\varphi(0)}\right) = 2(e^{-t}-1)$,

Initially empty system: $E_m(0) = 1$, which gives $\varphi(0) = 1$

$\varphi(t) = e^{[2(e^{-t}-1)]}$,

$$\boxed{E_m(t) = e^{-(m-1)t}\, e^{[2(e^{-t}-1)]}}$$

asymptotically ✓ — all larger gaps decay faster

Finally, $p(t) = 1 - E_1(t) = 1 - e^{[2(e^{-t}-1)]} \xrightarrow[t\to\infty]{} 1 - e^{-2}$.

Consider

$$p_\infty - p(t) = (1-e^{-2}) - (1-e^{2(e^{-t}-1)}) =$$

$$= e^{-2}\underbrace{(\underbrace{1-e^{2e^{-t}}}_{\sim 2e^{-t}})}_{\sim e^{-t}(2e^{-2})} \to \emptyset \text{ as } t\to+\infty$$

Exponential approach to the asymptotic value

x.: trimers, etc.

$$e^{2(e^{-t}-1)} - e^{-2} = e^{-2}(\underbrace{e^{2e^{-t}}}_{\geq 1}-1)$$

Redner, Lecture 5

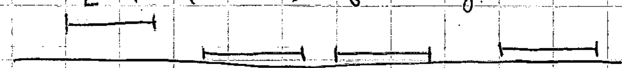
Dimer deposition: $p(t) = 1 - e^{-2(e^{-t}-1)} \xrightarrow[t\to\infty]{} 1-e^{-2} \simeq 0.866...$

SS approached exponentially $\rightarrow$ $p_\infty - p(t) \sim e^{-t}$

Ex. K-mer deposition, do by same techniques as dimers!

Now, cars: $L=1$ ("cars of length 1")

(1D)



"Cars are parked & abandoned"

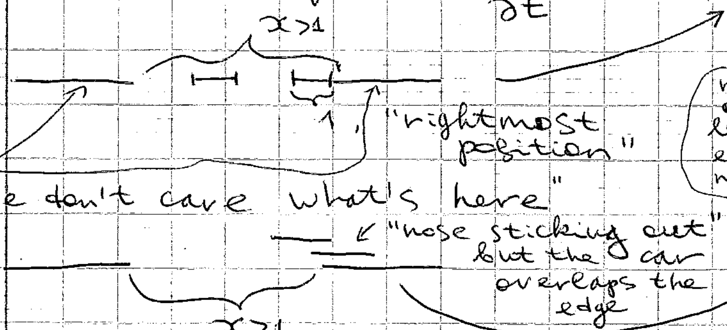
Renyi: $p_\infty \simeq 0.747...$

$E_m(t) \to E(x,t)$ = prob. that $\exists$ an empty interval of length x, & don't care about outside of the interval

$E_m(t)$: discrete case (e.g. for dimers); $E(x,t)$: contin. case

Evol'n eq'n: $$\frac{\partial E(x,t)}{\partial t} = -(x-1)E(x,t) - 2\int_x^{x+1} E(y,t)\,dy \quad \boxed{x>1}$$

right and left edge need to be counted



1st term, RHS: # ways in which $x<1$ interval can be completely covered by a car

2nd term, RHS: ~~# ways in which $x<1$ interval is~~ partially covered by a car

Note that $\frac{\partial E(0,t)}{\partial t} = -E(1,t)$

Likewise, $$\frac{\partial E(x,t)}{\partial t} = -(1-x)E(1,t) - 2\int_1^{x+1} E(y,t)\,dy \quad \boxed{x<1}$$

(Ex.) $x \leftrightarrow 1$ everywhere

Try $E(x,t) = \underbrace{e^{-(x-1)t}}_{\text{sol'n of the homogen. eq'n}}\underbrace{\varphi(t)}_{\text{unknown f'n}} = e^{-(x-1)t}E(1,t)$

So, $\dot\varphi e^{-(x-1)t} = -2\varphi\int_x^{x+1} e^{-(y-1)t}\,dy$, for $x>1$:

$$\dot\varphi = +\frac{2\varphi}{t} e^{xt}\left[e^{-(x+1)t} - e^{-xt}\right] = -\frac{2\varphi}{t}(1-e^{-t}).$$

So, $\underbrace{\varphi(t)}_{E(1,t)} = e^{-\int_0^t \frac{2(1-e^{-u})}{u}du}$

Consider $p(t) = 1 - E(0,t)$ occupancy ("density of occupied space")

$$\frac{\partial E(0,t)}{\partial t} = -E(1,t) \quad \text{[see above]}$$

Then $\underbrace{E(0,t)}_{1-p} - \underbrace{E(0,0)}_{1} = -\int_0^t E_1(t')dt'$, or

$$p(t) = \int_0^t E_1(t')dt',$$

$$p(t) = \int_0^t \exp\left[-\int_0^v \frac{2(1-e^{-u})}{u}du\right]dv.$$

Numerically: $p_\infty \cong 0.747\ldots$ slightly less effic. than dimers

$$p_\infty - p(t) = \int_t^\infty dv\, e^{-\int_0^v 2\left(\frac{1-e^{-u}}{u}\right)du} \underset{t\to\infty}{\sim}$$

$$\int_0^v \frac{1-e^{-u}}{u}du \simeq \log v + \gamma_E + \frac{e^{-v}}{v} + \ldots$$

$$\sim \int_t^\infty dv \exp[-2(\log v + \gamma_E)] \to \frac{e^{-2\gamma_E}}{t}.$$

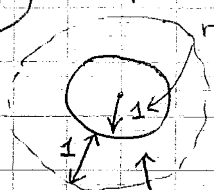
γ_E: Euler const 0.577

$\sim \frac{1}{t}$ relaxation

interlu

"Spaces that are infinitesimally longer than a car arise and are filled extremely slowly"

D>1 disk adsorption

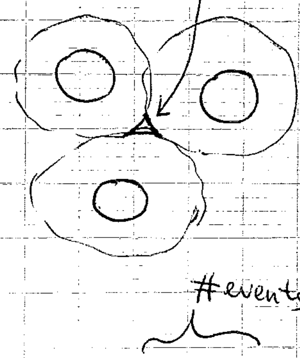


exclusion zone, cannot put the center of another disk there.

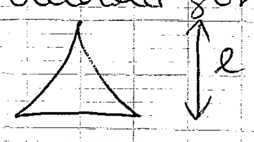
Exclusion zones can overlap, disks cannot

Target zone: complement of the union of all excluded zones

allowed zone, here you can put another disk center



Allowed zone:



area $\sim \ell^2$

$c(\ell,t)$ = density of allowed zones of linear scale ℓ.

$$\frac{\partial c}{\partial t} = -\underset{\text{density}}{c}\,\overbrace{\ell^2}^{\text{\#events}}_{\text{area of the zone}} \Rightarrow c(\ell,t) \sim \exp[-\ell^2 t]$$

adsorption rate ↙ upper cutoff not strictly ∞ but the $\int$ is vanishingly sm[...]

$$\overbrace{\frac{\partial p}{\partial t}}^{\text{adsorption rate}} = \underbrace{\int_0^{\infty} c\,\ell^2 \, d\ell}_{\substack{\text{area fraction}\\ \text{of the target zones}}} = \int e^{-\ell^2 t}(\ell^2 t)\, d(\ell\sqrt{t})\, \frac{1}{t^{3/}}$$

$$x = \ell\sqrt{t} \qquad \int e^{-x^2} x^2\, dx\, \frac{1}{t^{3/2}}$$

So, $\frac{\partial p}{\partial t} \sim t^{-3/2}$

jamming exponent $\sigma = \frac{1}{2}$

$$p_\infty - p(t) \sim t^{-1/2} \qquad \boxed{D=2}$$

general D: $p_\infty - p(t) \sim t^{\overbrace{-1/D}^{\sigma}}$, general D

Coarsening

Comput'n: 3D cube of spins (periodic BCs), run MC (Metro[...] simul'n @ $T=0$ ⇒ "never" reaches the ground state

Move set: single-spin flip; no magnetic field

(w/ magnetic field, ~~do~~ the ener[...] trivially goes to $\emptyset$ [energy is defined as fraction of "incorrect" bonds

interlude

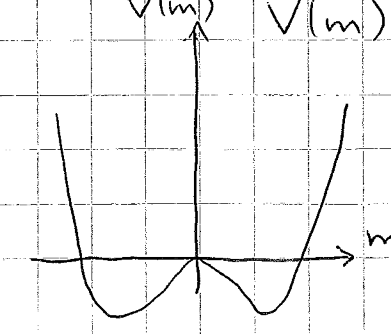
Time-dependent Ginzburg-Landau eq. (TDGL) [continuous mode[...]

$$F\{m(x)\} = \int dx \left[\underbrace{|\nabla m(x)|^2}_{} + V(m(x)) \right]$$

continuum descript'n for misaligned spins, $|\ldots|^2$ due t[o] symmetry re[...]

$V(m)$ should favor $m = \pm 1$: choose

$$V(m) = \frac{1}{2}(1-m^2)^2$$



This is a phenomenological model; $m(x)$ → magnetization @ point

Complex networks

Erdos - Renyi graph (ER)

N points

→ $G(N,L)$ N vertices, L links

→ $G(N,p)$ N vertices, each link occupied w/prob. P/N

Degree of a node (d) ⟹ its # links

$$\langle d\rangle = \frac{2L}{N}$$, b/c each link connects 2 nodes

→ "time-dependent" version: $G(N,r)$

N vertices, introduce links at rate r: $L = rt$; if $r = \frac{N}{2} \Rightarrow \langle d\rangle = t$

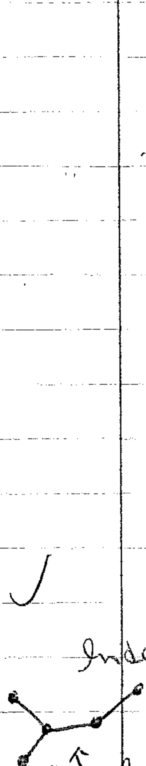
$n_k \equiv$ fraction of nodes w/degree k

$$\dot n_k = n_{k-1} - n_k \Rightarrow n_k = \frac{t^k}{k!}e^{-t}$$

Indeed, $\dot n_k = \underbrace{\frac{k t^{k-1}}{k!}e^{-t}}_{n_{k-1}} - n_k$

↳ Poisson distr'n

$n_0(t) = e^{-t}$, $n_0(0) = 1$



cluster w/5 nodes

$C_k \equiv$ fraction of clusters w/k vertices =

$$= \frac{k^{k-2}}{k!}t^{k-1}e^{-kt}$$

just like an aggreg'n problem w/product kernel (prob. of joining clusters i & j ~ ij)

Prob. to join 2 clusters of size i & j: (c_i = # clusters of size i)

$$\left(\frac{i c_i}{N}\right)\left(\frac{j c_j}{N}\right) = i j c_i c_j$$

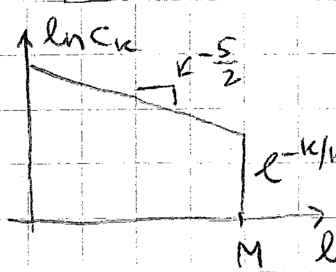
($c_i c_j$: concentr'n of clusters)

$$C_k \approx \frac{1}{\sqrt{2\pi}}\frac{1}{k^{5/2}}e^{-k(1-t)^2/2}$$

$t \to 1^-$

So,

$$\dot c_k = \frac{1}{2}\sum_{i+j=k} i c_i j c_j - k c_k$$



e^{-k/k^*} $k^* = \frac{2}{(1-t)^{-2}}$

Percolation transition at $t=1$: mean cluster size finite at $t<1$ but diverges at $t=1$ in an infinite network

↑ max cluster

Max. size M: $N\int_M^\infty C_k\, dk = 1$ ← "$O(1)$", only 1 large cluster

$t<1$ $N\int_M^\infty \frac{1}{k^{*5/2}}e^{-k/k^*}dk = 1$

$\frac{N}{k^{*5/2}}e^{-M/k^*} = 1 \Rightarrow M = k^* \log N$

$k^* \log\left(\frac{N}{k^{*5/2}}\right) = M$, $M \sim \log N$