What are the general lessons of the RG approach?

Consider \( \tilde{\mathcal{H}} = \mathcal{H} / k_b T \) — reduced Hamiltonian

**RG transform:** \( \tilde{\mathcal{H}}' = \hat{R} \tilde{\mathcal{H}} \)

RG operator, decreases the # of d.o.f. from \( N \to N' \)

(e.g. by grouping spins)

Then \( \hat{b}^d = \frac{N}{N'} > 1 \)

scale factor of the RG transform

Essential condition: \( \mathcal{Z}_{N'}(\tilde{\mathcal{H}}') = \mathcal{Z}_N(\tilde{\mathcal{H}}) \)

Then reduced free en. per spin is

\( \tilde{f}(\tilde{\mathcal{H}}') = \hat{b}^d \tilde{f}(\tilde{\mathcal{H}}) \)

Likewise, \( \hat{\mathcal{V}}' = \hat{b}^{-1} \hat{\mathcal{V}} \)

\( \tilde{q}' = \hat{b} \tilde{q} \), etc.

Note also that at fixed points,

\( \tilde{\mathcal{H}}' = \tilde{\mathcal{H}} \equiv \tilde{\mathcal{H}}^* \)

Also, \( \hat{b}' = \hat{b}^{-1} \hat{b} \)

But we expect self-similarity:

\( \hat{b}' = \hat{b} = \hat{b}^* \Rightarrow \hat{b}^* = \infty \)

at criticality.
Flows in primal space

Consider \( \tilde{H}(\tilde{\mu}) \)

vector of parameters like \( J, H \) (or \( K, h \))

Under RG transform,

\[
\tilde{\mu}' = \hat{R} \tilde{\mu} \quad (*)
\]

At a fixed point, \( \tilde{\mu}' = \tilde{\mu}^* = \hat{R} \tilde{\mu}^* \)

Close to a fixed point,

\[
\begin{align*}
\tilde{\mu} &= \tilde{\mu}^* + \delta \tilde{\mu}, \\
\tilde{\mu}' &= \tilde{\mu}^* + \delta \tilde{\mu}'
\end{align*}
\]

Indeed, \((*)\) gives

\[
\tilde{\mu}^* + \delta \tilde{\mu}' = \hat{R} (\tilde{\mu}^* + \delta \tilde{\mu})
\]

Taylor expansion of \( \hat{R} \)

\[
\hat{R} \tilde{\mu}^* = \tilde{\mu}^*
\]

\( \hat{R} \to \) linear transform

\( A \): eigenvalues \( \lambda_i \),

eigenvectors \( \vec{\Theta}_i \)

Under RG,

Consider 2 transforms:

\[
A_1 A_2 = U \Phi \left( \begin{array}{cc}
\lambda_1(k) & 0 \\
0 & \lambda_2(k)
\end{array} \right) U^{-1} U \left( \begin{array}{cc}
\lambda_1(k) & 0 \\
0 & \lambda_2(k)
\end{array} \right) U^{-1}
\]

2x2 matrices, for example

\[
= U \left( \begin{array}{cc}
\lambda_1(k) \lambda_1(k) & 0 \\
0 & \lambda_2(k) \lambda_2(k)
\end{array} \right) U^{-1}
\]

On the other hand,

\[
A_1 A_2 = A = U \left( \begin{array}{cc}
\lambda_1(k^2) & 0 \\
0 & \lambda_2(k^2)
\end{array} \right) U^{-1}
\]

\( U \) is a matrix with \( \vec{\Theta}_i \) as columns: \( U = [\vec{u}_1 \; \vec{u}_2] \)
So, \( \lambda_i(b) \Lambda_i(b) = \lambda_i(b^2) \)

\[
\frac{1}{\lambda_i(b)} = b y_i \quad y_i: \text{indep. of } b
\]

Thus, near a fixed point

\[
\bar{\mu} = \bar{\mu}^* + \sum g_i \bar{\phi}_i \quad \text{expansion in terms of eigenvectors of } A
\]

\( R \)G transform:

\[
\bar{\mu}' = \bar{\mu}^* + A(\bar{\mu}^*) \sum g_i \bar{\phi}_i = \bar{\mu}^* + \sum g_i b y_i \bar{\phi}_i
\]

\[
A(\bar{\mu}^*) \bar{\phi}_i = \lambda_i \bar{\phi}_i
\]

So, \( g'_i = b y_i - y_i \)

\( y_i > 0 \) => the system is driven away from the fixed point =>

=> relevant variable \( i \)

\( y_i < 0 \) => the system moves closer to the fixed point => irrelevant var.

\( y_i = 0 \) => marginal var., 1st order expansion insufficient

Thus, each fixed point is characterized by relevant & irrelevant vars. (scaling fields)
Consider flow lines with $g_2 \neq 0$.

Let's say

\[ \begin{cases} 
  g_1 \text{ is irrelevant,} \\
  g_2 \text{ is relevant} 
\end{cases} \]

\[ g_2 = 0 \] is the critical surface (at least locally)

Critical surface:
all points that flow into a critical point

All points on the crit. surf. have $\infty \%$, since the fixed point has $\infty \%$. RG transforms can only decrease $\%$.

Note that all systems that flow close to the critical point will be characterized by the same exponents $y_i \rightarrow \, \underbrace{\text{universality}}$.

Consider another example:

A: 1 relevant, 1 irrelevant field
B: 2 irrelevant fields

These get very close to A before being pushed to B $\rightarrow$ crossover
So the procedure is:

1) Write down renormalization flow eq's 
   e.g. \( \{ x' = f(x, y) \} \)
   \( \{ y' = g(x, y) \} \)

2) Find fixed points
   \( \{ x' = x = x^* \} \)
   \( \{ y' = y = y^* \} \)

3) Linearize around each fixed point:
   
   \[
   \begin{pmatrix}
   \delta x' \\
   \delta y'
   \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
   \begin{pmatrix} \delta x \\
   \delta y
   \end{pmatrix}
   \]

   Find \( a \) & \( b \) for \( A \)
   Find \( y_i \) \( \approx \) relevant/irrelevant fields

4) Find critical surfaces

Scaling & critical exponents

Recall that

\[
\tilde{f}(\tilde{\mu}) = \tilde{f}(\mu')
\]

Then \( \tilde{f}(y_1, y_2, y_3, \ldots) = \tilde{f}(b_{1} y_1, b_{2} y_2, b_{3} y_3, \ldots) \)

close to the fixed point

Thus \( \tilde{f} \) is a generalized homogeneous function:

\[
f(x^a, y^b) = \lambda f(x, y)
\]

e.g. \( f(x, y) = x^3 + y^3 \)

\[
\lambda f(x, y) = (x^{1/3} x)^3 + (x^{1/3} y)^3 = \Rightarrow \begin{cases} a = 1/3 \\ b = 1/3 \end{cases}
\]

Now recall that \( C_u \sim \left( \frac{\partial^2 \tilde{f}}{\partial t^2} \right)_{\mu = 0} \sim |t|^{-d} \)

at zero field

(Table 2.3)
Assume that \( g_1 = t \), \( g_2 = h \) are relevant & all others irrelevant, then

\[
\begin{align*}
\begin{cases}
t = \frac{T - T_c}{T_c}, \\
\frac{h}{k_b T}
\end{cases}
\end{align*}
\]

\[
\mathcal{f}(t, h) \sim e^{-d} \mathcal{f}(e^{y_1 t}, e^{y_1 h})
\]

Can set \( y_2 \to 0 \), \( y_1 \to 0 \), ... 

Then \( \left( \frac{\partial^2 \mathcal{f}}{\partial t^2} \right)_{h=0} \approx \left( 6 - d + 2y_1 \right) \mathcal{f}_{tt} \left( e^{y_1 t}, 0 \right) \)

\[
\mathcal{f}_{tt}(t, h=0)
\]

Now, choose \( b y_1 |t| = 1 \) : \( \beta = |t|^{-\frac{1}{y_1}} \)

\[
\mathcal{f}_{tt}(t, 0) \sim |t|^{-\frac{4}{y_1}} (2y_1 - d) \mathcal{f}_{tt}(\pm 1, 0) =
\]

\[
= |t|^{\frac{d - 2y_1}{y_1}} \mathcal{f}_{tt}(\pm 1, 0).
\]

So, \( \beta = 2 - \frac{d}{y_1} \)

Similarly, for zero-field magnetization

\[
M \sim (-t)^\beta \Rightarrow \beta = \frac{d - y_2}{y_1}
\]

Zero-field isothermal susceptibility

\[
X_T \sim |t|^{-\gamma} \Rightarrow \gamma = \frac{2y_2 - d}{y_1}
\]

But then \( d + 2\beta + \gamma = 2 - \frac{d}{y_1} + \frac{2d}{y_1} - \frac{2y_2}{y_1} + \frac{2y_2}{y_1} - \frac{d}{y_1} = 2 \)
Recall Rushbrooke inequality:
\[ 1 + 2\beta + \gamma \geq 2. \]

Turns out it's an equality (\(!\))

Note also that the exponents are the same below & above T_c

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Finally,

Consider \( M(t, h) \sim h^{d-y} \times M(b^{y_1}t, b^{y_2}h) \)
\[ M(t, h) \sim (\frac{2\xi}{\theta t})^t \]

Choose \( b^{y_1} |t| = 1 \):

\[ M(t, h) \sim |t|^{\frac{(d-y_2)}{y_1}} \times M(\pm 1, h |t|^{-\frac{y_2}{y_1}}) \]

where

\[ s = \frac{y_2}{d-y_2} \]

Critical isotherm:
\[ H \sim |M|^s \sinh(M) \]

\[ T = T_c \]

So, \( M(t, h) \sim |t|^{\beta} \times M(\pm 1, h |t|^{-\beta s}) \)

Define \( \bar{m} = \frac{M(t, h)}{|t|^{\beta}} \)

\[ \bar{m} \sim M(\pm 1, 1/h) \]

Collapse of the data on 2 curves (one below \( T_c \), one above \( T_c \)):

\[ \frac{1}{h} \leq \text{verified experimentally} \]