# Topics for L4

- Path integral + source terms

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-(\bar{\psi}M\psi - \bar{J}\psi - \bar{\psi}J)} = \frac{e^{\bar{J}M^{-1}J}}{\det M}$$

$$M = \partial_\tau + \mathcal{H} = -\mathcal{G}^{-1} \qquad Z = \frac{e^{-\bar{J}\mathcal{G}J}}{\det[-\mathcal{G}^{-1}]}$$

Multi-dimensional

$J$

$\bar{J}$

- Feynman Diagrams

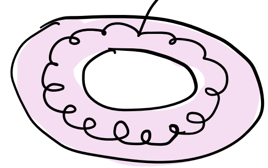
$F = F_0 + \ldots$

$\mathcal{G}_0 = \longrightarrow$

$\mathcal{G} = [\,\ldots\,]$

- Quick look at: Superfluids + BECs

$$\psi(x) = \sqrt{n_s}\, e^{i\phi(x)}$$



$$S = \iint \left[ \bar{\psi}\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi + \tfrac{1}{2} g (\bar{\psi}\psi)^2 \right] d\tau$$

superfluid / BEC

$$\oint \nabla\phi \, dx = 2\pi n$$

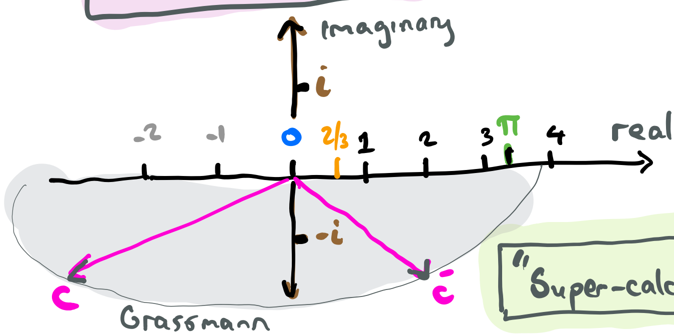
Topological invariant

$$|\psi(x)\rangle = \exp\left[\int d^3x \, \hat{\psi}^\dagger(x)\psi(x)\right]|0\rangle$$

$$\hat{\psi}(x)|\psi(x)\rangle = \psi(x)|\psi(x)\rangle$$

- Fermions!

$$|c\rangle = e^{\sum_\lambda \hat{c}^\dagger_\lambda c_\lambda}|0\rangle \qquad \hat{c}_\lambda|c\rangle = c_\lambda|c\rangle$$




"Super-calculus"

anticommutes

$$\hat{c}_\lambda \hat{c}_{\lambda'} = -\hat{c}_{\lambda'}\hat{c}_\lambda \Rightarrow c_\lambda c_{\lambda'} = -c_{\lambda'}c_\lambda$$

$$\hat{c}_\lambda^2 = 0 \Rightarrow c_\lambda^2 = 0$$

$$\partial_x x^n = n x^{n-1} \qquad \int dx\, x^n = \frac{x^{n+1}}{n}$$

$$\partial_x e^{xa} = a e^{ax} \;\ldots \qquad \int dx \frac{1}{x} = \ln x \;\ldots$$

$$\partial_x \ln x = 1/x$$

$\vdots$

$$\partial_c 1 = 0 \qquad \partial_c = \int dc \qquad \int dc\, 1 = 0$$

$$\partial_c c = 1 \qquad \int dc\, c = 1$$

GRASSMANN CALCULUS

$$\partial_c F[c] \overset{!}{=} \int dc\, F[c]$$

# Path Integrals with Sources

$$\int \mathcal{D}[\bar{b},b]\, e^{-(\bar{b}Mb - \bar{j}b - \bar{b}j)} = \frac{e^{\bar{j}Mj}}{\det M}$$

First prove $\displaystyle \int \mathcal{D}[\bar{b},b]\, e^{-\bar{b}Mb} = \frac{1}{\det M}$ $\qquad b = (b_1, \dots b_N)$

Let $m_\lambda$ be eigenvalues of $M$ : $\quad M_{rs} U_s^{\lambda} = U_r^{\lambda} m_\lambda = \begin{pmatrix} \cdots & \vec{u}^{\lambda} & \cdots \end{pmatrix}\begin{pmatrix} \ddots & & 0 \\ & m_\lambda & \\ 0 & & \ddots \end{pmatrix}$

$\Rightarrow u^{+} M u = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_N \end{pmatrix}$

Let $b = Ua \Rightarrow \quad \bar{b}\underline{M}b = \bar{a}\underline{U}^{+}\underline{M}\underline{U}a = \sum \bar{a}_\lambda m_\lambda a_\lambda$

$$\int \mathcal{D}[\bar{b},b]\, e^{-\bar{b}Mb} = \int \prod \frac{d\bar{a}_\lambda\, da_\lambda}{2\pi i}\, e^{-\sum \bar{a}_\lambda m_\lambda a_\lambda} = \frac{1}{\prod_\lambda m_\lambda} = \frac{1}{\det M}$$

Now shift $\quad b \to b - M^{-1}j \quad , \quad \bar{b} \to \bar{b} - \bar{j}M^{-1}$

$$\int \mathcal{D}[\bar{b},b]\, e^{-\bar{b}Mb} = \int \mathcal{D}[\bar{b},b]\, e^{-(\bar{b}-\bar{j}M^{-1})M(b-M^{-1}j)}$$

$$= e^{-\bar{j}M^{-1}j} \int \mathcal{D}[\bar{b},b]\, e^{-(\bar{b}Mb - \bar{j}b - \bar{b}j)}$$

$$= \frac{1}{\det M}$$

$$\Rightarrow \int \mathcal{D}[\bar{b},b]\, e^{-(\bar{b}Mb - \bar{j}b - \bar{b}j)} = \frac{e^{\bar{j}M^{-1}j}}{\det(M)}$$

Non interacting bosons $\quad M = -G^{-1} = (\partial_\tau + \underset{\sim}{H})$

# Wick's Thm.

$$\underbrace{Z[\bar{j},j]}_{} = \int \mathcal{D}[\bar{b},b]\, e^{-\overbrace{\int\left(\bar{b}(-g^{-1})b - \bar{j}b - \bar{b}j\right)}^{S}} = \frac{e^{-\int \bar{j}(1)\, g(1,2)\, j(2)\, d1\, d2}}{\det[-g^{-1}]}$$

$$= \mathrm{Tr}\left[T e^{-\int(h - \bar{j}\hat{b} - \hat{b}^{+}j)d\tau}\right]$$

Functional
= Function of function

Free particles in arbitrary source fields

$$\frac{\delta}{\delta j(1)} \equiv b(1) \qquad \frac{\delta}{\delta \bar{j}(1)} \equiv b^{+}(1)$$

$$\langle T b(2) \rangle = \frac{\delta}{\delta \bar{j}(2)} \ln Z[\bar{j},j] = -\int g(2-1)\, j(1)\, d2$$

$$\langle T b(2)\, \bar{b}(1) \rangle = \frac{\delta}{\delta j(1)} \frac{\delta \ln Z}{\delta \bar{j}(2)} = -g(2-1)$$

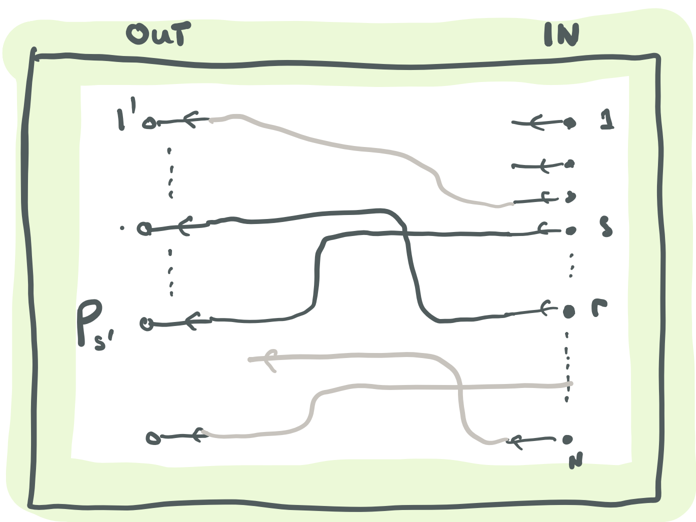
$$\Rightarrow -\langle T b(2) \bar{b}(1) \rangle = g(2-1)$$

Consider

$$G(1' \dots n' ; n \dots 1) = (-)^{r} \langle T b(1') \dots b(n')\, b^{+}(n) \dots b^{+}(1) \rangle$$

$$n' \equiv (\vec{x}_n', t_n')$$

$$= \frac{(-1)^r}{Z} \int \mathcal{D}[\bar{b},b]\, e^{-S}\, b(1') \dots \bar{b}(1)$$




$$= (-1)^r \frac{\delta^{2r}}{\delta \bar{j}(1') \dots \delta \bar{j}(n') \delta j(n) \dots \delta j(1)} \ln Z \Bigg|_{j,\bar{j}=0}$$

$\delta[s' - P_r']$

$$= \frac{\delta^{r}}{\delta \bar{j}(1') \dots \delta \bar{j}(n')} \prod_{s=1}^{n} \int ds'\, \bar{j}(s')\, g(s',r)$$

$$\boxed{= \sum_{P} \prod_{r=1}^{n} G(P_{r'}, r)}$$

"contraction"

$$= (-1)^r \langle T \dots \overbrace{\hat{b}(P_r') \dots \hat{b}^{+}(r)}^{-G(P_r',r)} \dots \rangle$$

Particles are identical ∴
All possible permutations possible

Wick's Thm. BASIS FOR CONVENTIONAL FEYNMAN DIAGRAMS.

$$1 \longrightarrow 2 = g(2,1)$$

$$g_0(p, i\omega_n) = \frac{1}{i\omega_n - \omega_p} = \frac{1}{-\partial_\tau - h}$$

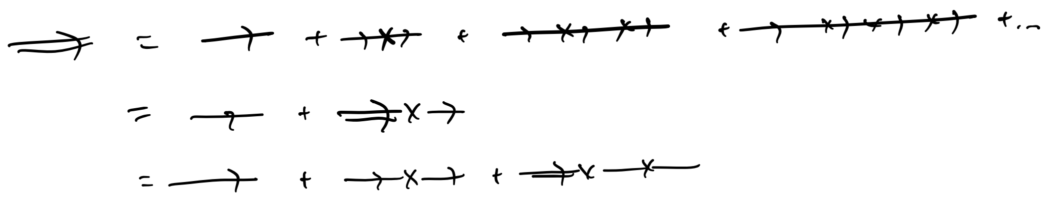
$$\omega_p \to \omega_p + V \qquad g \to \frac{1}{i\omega_n - \omega_p - V} = \frac{1}{(i\omega_n - \omega_p)\left(1 - \frac{V}{i\omega_n - \omega_p}\right)} = \frac{G_0}{1 - G_0 V}$$

# FEYNMAN DIAGRAMS

Artist $G = -\frac{1}{\partial_\tau + h}$

Engineer $G_{\lambda\lambda'} = \frac{\delta_{\lambda\lambda'}}{i\nu_n - \omega_\lambda}$

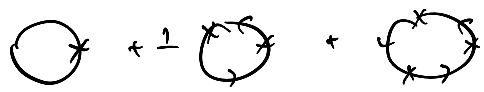
$= G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \dots$



Free - energy

$$F = F_0 + T \sum \ln\left(1 - G_0 V\right)$$

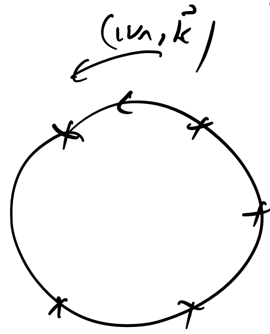
$$= F_0 - \frac{1}{\beta} \sum_{kn} G_0 V + \frac{1}{2}(G_0 V)^2 + \frac{1}{3}(G_0 V)^3$$

$= F_0 - \frac{1}{\beta} \Big[$  $+ \dots \Big]$

$$= F_0 - T \sum_{k,n} \left[ \frac{1}{i\nu_n - \omega_k} V + \frac{1}{2}\left(\frac{V}{i\nu_n - \omega_k}\right)^2 + \frac{1}{3}\left(\frac{V}{i\nu_n - \omega_k}\right)^3 + \dots \right.$$

 $\frac{1}{i\nu_n - \omega_k} = G_0$

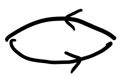
Sum of linked diagrams




 $V$

$\frac{1}{n}$ symmetry fcts.

Closed loops Sum over all intermediate frequencies & momenta. $T \sum_{n,k}$

$-\frac{\partial^2 F}{\partial V_{q=0}^2} =$  $= T \sum \frac{1}{(i\nu_n - \omega_k)^2}$