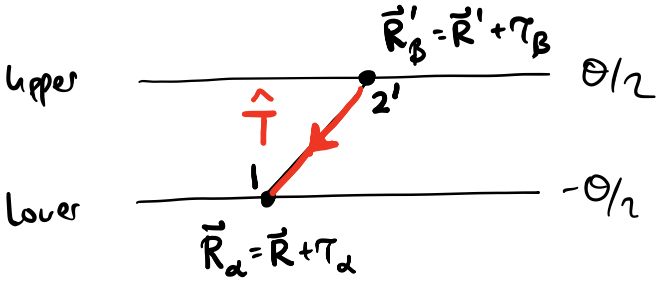
L24 : BM Model - Interlayer tunnelling.



$t\,(\vec R + \tau_\alpha - \vec R' - \tau'_\beta)$ = tunneling amplitude

$$= f\left(\sqrt{(\vec R - \vec R' + \tau_\alpha - \tau'_\beta)^2 + d^2}\right)$$

$$|\psi^{(1)}_{\vec k \alpha}\rangle = \frac{1}{\sqrt{N_s}} \sum_{\vec R} e^{i\vec k\cdot(\vec R + \vec \tau_\alpha)} |R + \tau_\alpha\rangle$$

$\uparrow$ sublattice

$$|\psi^{(2)}_{\vec p \beta}\rangle = \frac{1}{\sqrt{N_s}} \sum_{\vec R'} e^{i\vec p\cdot(\vec R' + \vec \tau'_\beta)} |\vec R' + \tau'_\beta\rangle$$

} Bloch States on the lower + upper Graphene Layers.

$$T^{\alpha\beta}_{kp'} = \langle \psi^{(1)}_{k\alpha} | H_T | \psi^{(2)}_{\vec p'\beta} \rangle$$

$$H_T = \sum_{\substack{R\,\tau_\alpha \\ \vec R',\tau'_\beta}} |R + \tau_\alpha\rangle\, t(\vec R + \vec\tau_\alpha - \vec R' - \vec\tau'_\beta)\, \langle \vec R' + \tau'_\beta|$$

$$\Rightarrow \boxed{T^{\alpha\beta}_{kp} = \frac{1}{N_s} \sum_{\substack{R\tau_\alpha \\ \vec R'\tau'_\beta}} e^{-i\vec k(\vec R + \vec\tau_\alpha)}\, t(R + \tau_\alpha - \vec R' - \tau'_\beta)\, e^{i\vec p'\cdot(R' + \tau'_\beta)}}$$

$$t_{\vec q} = \int d^2r\, t(r)\, e^{-i\vec q\cdot\vec r} \Longleftrightarrow t(\vec r) = \int \frac{d^2 q}{(2\pi)^2}\, t_q\, e^{i\vec q\cdot\vec r}$$

$$t(\vec R) = \frac{1}{V} \sum_q t_q e^{i\vec q\cdot\vec R} = \frac{1}{N\Omega} \sum t_q e^{i\vec q\cdot\vec R}.$$

$$t\,(\vec R_\alpha - \vec R'_\beta) = \frac{1}{N\Omega} \sum_q t_{\vec q}\, e^{i\vec q(\vec R_\alpha - \vec R'_\beta)}$$

$$\Rightarrow \quad T^{\alpha\beta}_{kp'} = \frac{1}{N^2\Omega} \sum_{\substack{\vec{R}_1\tau_\alpha \\ \vec{R}_2'\tau_\beta' \\ \vec{q}}} e^{-i(k-\vec{q})(R_1+\tau_\beta)} \, e^{i(\vec{p}'-q)(R_2'+\tau_\beta')} \, t_{\vec{q}}$$

$$\vec{R}_2'+\tau_\beta' = M(\vec{R}_2+\tau_\beta)+d$$

TAKE $d \to 0$

$$= \frac{1}{N^2\Omega} \underbrace{\sum_{\vec{R}_1} e^{-i(k-q)(R_1+\tau_\alpha)}}_{N\,\delta_{k-q,-G_1}\, e^{i\vec{G}_1\cdot\vec{\tau}_\alpha}} \underbrace{\sum_{R_2} e^{i(p'-q)\left[M(R_2+\tau_\beta)\right]}}_{N\,\delta_{\vec{p}'-\vec{q},-G_2'}\, e^{-iG_2\tau_B}\, e^{-iG_2'\cdot\vec{d}}} \, t_{\vec{q}}$$

$$\vec{q} = \vec{p}'+\vec{G}_2'$$

$$M^{-1}(\vec{p}'-\vec{q}) = M^{-1}(-\vec{G}_2') = -G_2$$

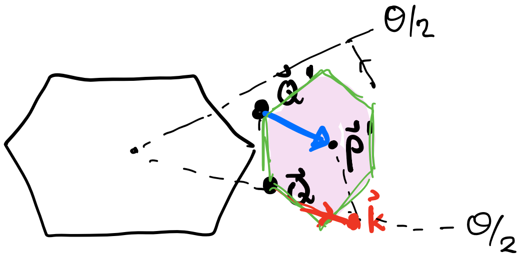
$$T_{\vec{k}\vec{p}'} = \frac{1}{\Omega} \sum_{G_1,G_2} e^{i(G_1\cdot\tau_\alpha - G_2\tau_\beta)} \, t_{\vec{k}+\vec{G}_1} \, \delta_{\vec{k}+\vec{G}_1,\vec{p}'+\vec{G}_2'}$$

$$\vec{p}'+\vec{G}_2' = \vec{k}+\vec{G}_1$$

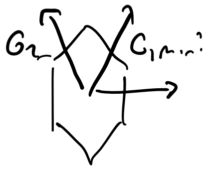
$$\Rightarrow \vec{p}' = k+\vec{G}_1-\vec{G}_2'$$

$$\underset{\sim}{M}\vec{G}_2 = \vec{G}_2'$$

What does this mean?

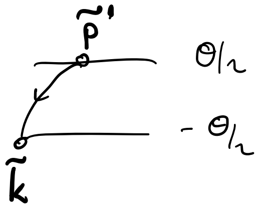


Let us measure momenta relative to the Dirac cones in the mini-BZ.



$$\tilde{p}' = \vec{p}'-\vec{Q}' = (k-Q') + \vec{G}_1-\vec{G}_2'$$

$$= k-Q+(Q-Q') + \underbrace{G_1-G_2'}_{G_{mini}}$$



$= k-Q + \vec{q}_0$

$=(\tilde{k} + q_0)$

$$\vec{q}_0 = (\vec{Q} - \vec{Q}') + \overbrace{(G_1 - MG_2)}^{\text{mini BZ}}$$

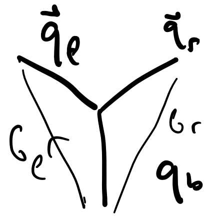
$$= \vec{q}_a + \vec{G}_{mini} - \underbrace{M(G_2 - G_1)}_{\to 0}$$

$$= \begin{cases} q_b = q_0 \\ q_r = q_1 \\ q_\ell = q_2 \end{cases}$$

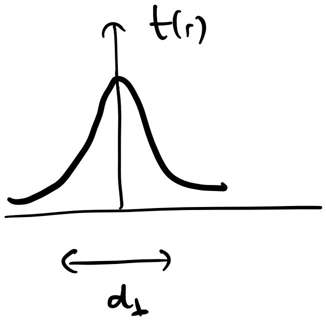
$\vec{q}_0 = (Q - Q') + G_{mini}$

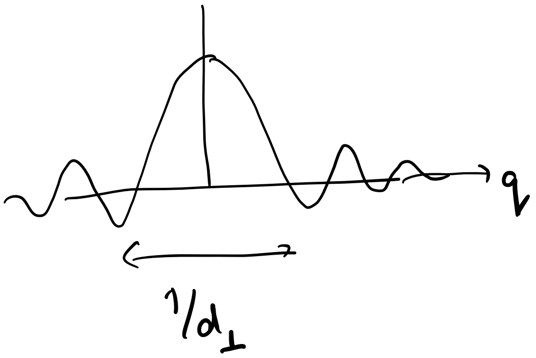
$= (Q-Q') + (G_1 - MG_2)$

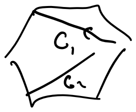
$G_{mini} = G_1 - MG_2$



$q_b = k_\theta(0,-1)$







$G \sim \frac{4\pi}{3a}$

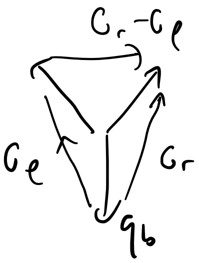
unless $G_2 \neq G_1$

Now since the range over which $t(q)$ is large is $q \lesssim \frac{1}{d_\perp} \ll G$

$G = 4\pi$

$R \sim \sqrt{d_\perp^2 + r^2}$

$T(q_b) = T(q_{tr}) = T(q_{t\ell}) = W$



$q_b = k_\theta(0,-1)$

$q_{tr} = k_\theta\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$q_{t\ell} = k_\theta\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$

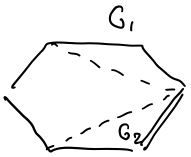
$G_\ell = k_\theta\left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

$G_r = k_\theta\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

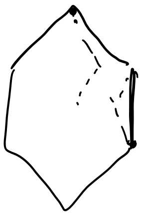
$G_r - G_\ell = k_\theta(3\sqrt{3}, 0)$

$G_1 = G_2$

$(G_1 \tau_\alpha - G_2 \tau_\beta) \longrightarrow G_2(\tau_\alpha - \tau_\beta)$

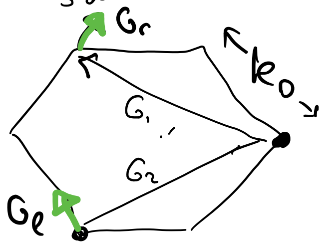


$$T_{kp}^{\alpha\beta} = W \quad e^{i \vec{G}_{\vec{M}}(\tau_\alpha - \tau_\beta)}$$



$$|G_2| = \sqrt{\frac{9}{4} + 3}\, k_D = \sqrt{3}\, k_D$$
$$= \frac{4\pi}{3a}$$

$$e^{i\ell 2\pi/3}$$



$$k_D = \frac{4\pi}{3\sqrt{3}a}$$

$$G_M = \begin{cases} 0 = G_1 \\ k_D\left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right) = G_2 \\ k_D\left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right) = G_3 \end{cases}$$

$$\vec{\tau}_2 = a(0,1)$$
$$\vec{\tau}_1 = \vec{0}$$

$$R_1 = \left(\frac{-\sqrt{3}}{2}, \frac{3}{2}\right)a$$

$$R_1 G_1 = 2\pi = k_D a\left(\frac{3\sqrt{3}}{2}\right)$$

$$\therefore k_D = \frac{4\pi}{a 3\sqrt{3}}$$

$$k_D a = \frac{4\pi}{3\sqrt{3}}.$$

$$G_M \cdot \tau = \begin{cases} 0 \\ \frac{4\pi}{3\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right) = 2\pi/3 \\ -2\pi/3. \end{cases}$$

$$T_b = W\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$T_\ell = W\begin{array}{c} \begin{array}{cc} 11 & 12 \end{array} \\ \begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix} \end{array}$$

$$"T_{q\ell} = W\begin{pmatrix} 1 & e^{i\phi\ell} \\ e^{-i\phi_\ell} & 1 \end{pmatrix}"$$

$$T \quad T^{\alpha\beta}(\quad) \quad W\begin{pmatrix} 1 & e^{i\phi} \end{pmatrix} = W\begin{pmatrix} 1 & e^{-2i\phi} \end{pmatrix}$$

$$T_r = T_{kp'}(G_3) = \omega \begin{pmatrix} & \\ e^{-i\phi} & 1 \end{pmatrix} = \omega \begin{pmatrix} & \\ e^{2i\phi} & 1 \end{pmatrix}$$

Can invert

$$T^{\alpha\beta}(\vec{r},\vec{r}') = \langle r|k\rangle T^{\alpha\beta}_{kp'} \langle p'|r'\rangle$$

$$= \frac{1}{\Omega_0} \sum_{k,p} e^{i(\vec{k}.\vec{r} - \vec{p}' r')} \delta_{\vec{k}+\vec{q}_e, \vec{p}'} T_e^{\alpha\beta}$$

$$= \frac{1}{\Omega_0} \sum e^{-i\vec{q}_e.\vec{r}'} T_e^{\alpha\beta} \sum_k e^{i\vec{k}(\vec{r}-\vec{r}')}$$

$$= \delta_{\vec{r},\vec{r}'} \sum e^{-i\vec{q}_e.\vec{r}} T_e^{\alpha\beta}$$

$$\boxed{T^{\alpha\beta}(r) = \sum_e e^{-i\vec{q}_e.\vec{r}} T_e^{\alpha\beta}}$$

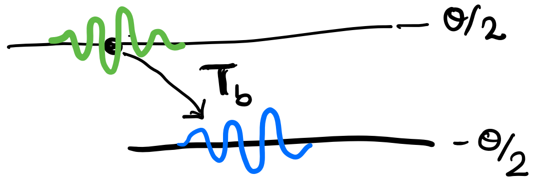
$q_e = -ik_\theta e^{ie\phi}$

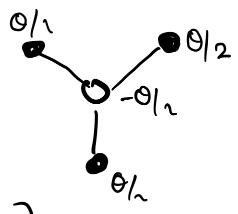
$\phi = \frac{2\pi}{3}$ $e=0,1,2$

The hopping from the lower to the upper is modulated by the wavevectors $q_e = -ik_\theta e^{ie\phi}$

BMM III

8 BAND MODEL + THE MAGIC ANGLE





$$H_k = \begin{bmatrix} h_k(-\theta/2) & T_0 & T_1 & T_2 \\ T_0^{+} & h_{k+q_b}(\theta/2) & & \\ T_1^{+} & & h_{k+q_r}(\theta/2) & \end{bmatrix}$$

$$\Big| \; T_2^{+} \qquad\qquad\qquad\qquad h_{k+q_\ell}(\theta/2) \Big]$$

$\vec{k}+\vec{q}_b$

$$h_k(-\theta/2) = v\begin{pmatrix} 0 & k^{*}e^{-i\theta/2} \\ k e^{i\theta/2} & 0 \end{pmatrix} \qquad h_{k+q_b}(\theta/2) = v\begin{pmatrix} 0 & (k+q_b)^{*}e^{i\theta/2} \\ (k+q_b)e^{-i\theta/2} & 0 \end{pmatrix}$$

$$G_{11}(\omega) = \omega - \left[h_k(-\theta/2) + \sum T_\ell \frac{1}{\omega - h_\ell} T_\ell^{+} \right]$$

$$h_\ell = v\begin{pmatrix} 0 & (k+q_\ell)^{*}e^{i\theta/2} \\ (k+q_\ell)e^{-i\theta/2} & 0 \end{pmatrix}\Bigg|_{k=0}$$

$$\frac{1}{A+b} = \frac{1}{A(1+A^{-1}b)}$$

$$= A^{-1} - A^{-1}bA$$

$$T_\ell \frac{1}{\omega - h_\ell} T_\ell^{+} \simeq T_\ell \frac{1}{-h_\ell\left(1 - \frac{\omega}{h_\ell}\right)} T_\ell^{+}$$

$$\simeq -T_\ell \frac{\omega}{h_\ell^2} T_\ell^{+} - T_\ell \frac{1}{v(k'+q_\ell)\cdot\vec{\sigma}} T_\ell^{+}$$

$$= -T_\ell \frac{\omega}{h_\ell^2} T_\ell^{+} - \Big(T_\ell h_\ell^{-1} T_\ell^{+} \underset{\overset{\uparrow}{\text{note}}}{\cancel{+}} T_\ell h_\ell^{-1} v(k_\ell'\cdot\sigma) h_\ell^{-1} T_\ell^{+}$$

$$h_\ell^{-1} = \frac{1}{(v_F k_\theta)^2}\begin{bmatrix} 0 & v\bar{k}e^{i\theta/2} \\ vke^{-i\theta/2} & 0 \end{bmatrix}$$

$$\left(\frac{W}{v_F k_\theta} = \alpha \right)$$

$$g(\omega) = t \quad \omega - h_{eff}$$

$$Z_\ell^{-1} = 1 + \frac{T_\ell T_\ell^\dagger}{(v_F k_\theta)^2} = \underline{1} + \alpha^2 \sum_\ell \begin{pmatrix} 1 & e^{-i\phi\ell} \\ e^{i\phi\ell} & 1 \end{pmatrix} \begin{pmatrix} 1 & e^{-i\phi\ell} \\ e^{i\phi\ell} & 1 \end{pmatrix}$$

$$= 1 + 6\alpha^2$$

$$\sum T_\ell h_\ell^{-1} T_\ell^\dagger = \alpha^2 \sum \begin{pmatrix} 1 & e^{-i\phi\ell} \\ e^{i\phi\ell} & 1 \end{pmatrix} \begin{pmatrix} 0 & \bar{q}_\ell \\ q_\ell & 0 \end{pmatrix} \begin{pmatrix} 1 & e^{-i\phi\ell} \\ e^{i\phi\ell} & 1 \end{pmatrix}$$

$$= \alpha^2 \sum_\ell \begin{pmatrix} e^{-i\phi\ell} q_\ell + h.c. & e^{-i\phi\ell} q_\ell + \bar{q}_\ell \\ q_\ell + e^{i\phi\ell} \bar{q}_\ell & e^{-i\phi\ell} q_\ell + h.c. \end{pmatrix} = 0 \, !$$

$$T_\ell h_\ell^{-1} v(k' \cdot \sigma) h_\ell^{-1} T_\ell^\dagger = ?$$

$$T_\ell h_\ell^{-1} = \frac{w}{v_F k_\theta} \begin{pmatrix} 1 & e^{-i\phi\ell} \\ e^{i\phi\ell} & 1 \end{pmatrix} \begin{pmatrix} & \overset{ie^{-i\phi\ell}}{\bar{q}_\ell / k_\theta} \\ \underbrace{q_\ell / k_\theta}_{-ie^{i\phi\ell}} & \end{pmatrix} = -i\alpha \begin{pmatrix} 1 & -e^{-i\phi\ell} \\ e^{i\phi_\ell} & -1 \end{pmatrix}$$

$$\parallel$$

$$T_\ell \frac{v(q_\ell \cdot \vec{\sigma})}{(v_F k_\theta)^2}$$

$$T_e h_e^{-1} v(k.\sigma) h_e T_e = \alpha^2 v \sum_e \begin{pmatrix} 1 & -e \\ e^{i\phi_e} & -1 \end{pmatrix} \begin{pmatrix} & \bar{k} \\ k & \end{pmatrix} \begin{pmatrix} 1 & e \\ -e^{i\phi_e} & -1 \end{pmatrix}$$

$$= v\alpha^2 \sum_e \begin{pmatrix} -e^{-i\phi_e} k & \bar{k} \\ -k & e^{i\phi_e} \bar{k} \end{pmatrix} \begin{pmatrix} 1 & e^{-i\phi_e} \\ -e^{i\phi_e} & -1 \end{pmatrix}$$

$$= v\alpha^2 \sum \begin{pmatrix} -e^{-i\phi_e} k - \bar{k} e^{i\phi_e} & -\bar{k} - k e^{-2i\phi_e} \\ -k - \bar{k} e^{2i\phi_e} & -k e^{-i\phi_e} - \bar{k} e^{i\phi_e} \end{pmatrix}$$

$$= v\alpha^2 \begin{pmatrix} 0 & -3\bar{k} \\ -3k & 0 \end{pmatrix}$$

$$h_{eff} = h_{eff}(-\theta/2) - \overbrace{T_e h_e^{-1}(\theta/2) T_e}^{\xrightarrow{\quad} \; = \left(T_e h_e^{-1} T_e^{+} \underset{\overset{\uparrow}{\text{note}}}{+} T_e h_e^{-1} v(k_e'.\sigma) h_e^{-1} T_e^{+} \right.}$$

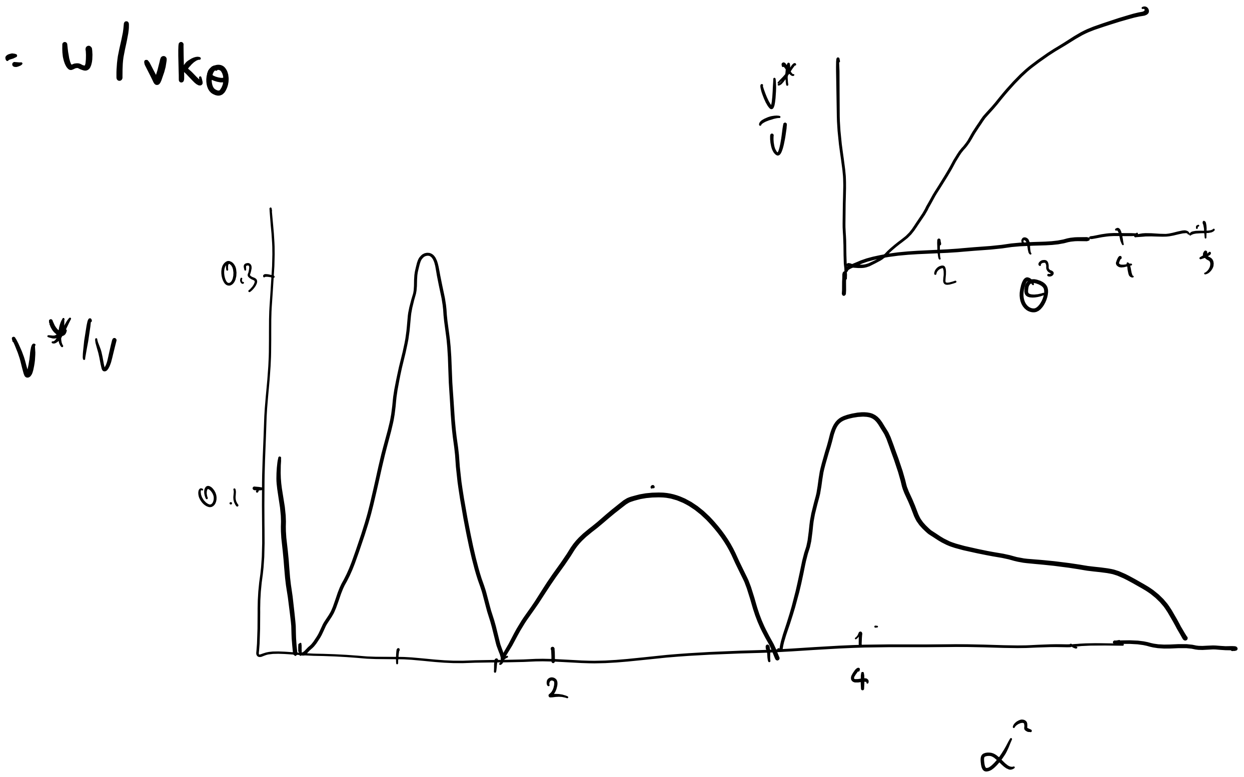
$$= v\left(1 - 3\alpha^2\right) \begin{pmatrix} 0 & \bar{k} \\ k & 0 \end{pmatrix}$$

$$\Rightarrow \mathcal{G}^{-1} = \left(1 + 6\alpha^2\right) \left[\omega - v^{*} \begin{pmatrix} 0 & \bar{k} \\ k & 0 \end{pmatrix} \right]$$

$$v^{*} = \frac{v\left(1 - 3\alpha^2\right)}{1 + 6\alpha^2}$$

Ignoring $k e^{\pm i\theta/2} \simeq k$

$\alpha = w / v k_\theta$



$W = 110$ meV

$v = 10^6$ m/s

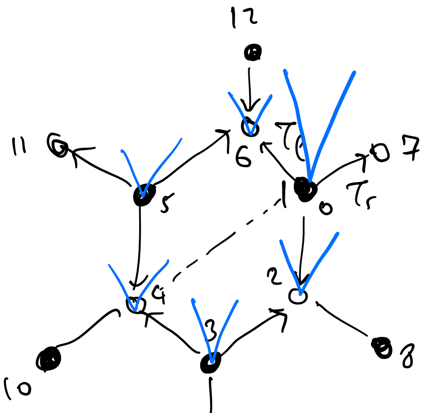
$a = 1.42 \times 10^{-10}$ m

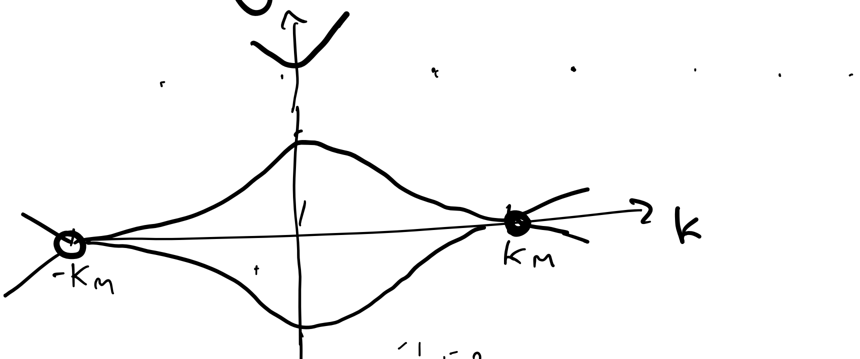
$k_0 = \frac{4\pi}{3\sqrt{3}a}$

$\alpha = \frac{1}{\sqrt{3}} = \frac{w}{v k_D \theta} \Rightarrow \theta = \frac{\sqrt{3}\, w}{v k_0}$

$= \underline{0.97^\circ}$

More accurately 1.05°.





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