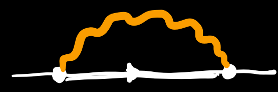
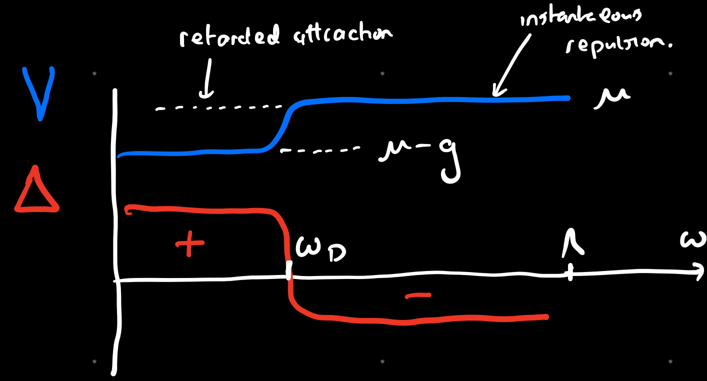
Retardation in Superconductivity

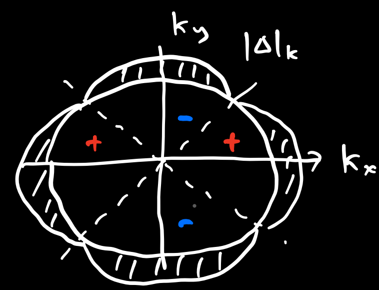


"Nobody has yet repealed Coulomb's Law"

Lev. Landau.

- Simple discussion.
- Migdal-Eliashberg Theory.





$\boxed{\text{Overcoming the Coulomb Interaction}}$

$$H = \sum \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \overbrace{\sum V_{\vec{k},\vec{k}'} (c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})(c_{-k'\downarrow} c_{k'\uparrow})}^{H_2}$$

Generalized BCS H.

$$H_I \longrightarrow \sum_k \left(\bar{\Delta}_{\vec{k}} (c_{-k\downarrow} c_{k\uparrow}) + (c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) \Delta_k \right) - \sum \bar{\Delta}_k V^{-1}_{k,k'} \Delta_{k'}$$

inverse of $V_{k,k'}$

$$E_{\vec{k}} = \sqrt{\epsilon_k^2 + |\Delta_k|^2}$$

$$F = -2T \sum_k \ln\left(2\cosh \frac{\beta E_k}{2}\right) - \sum \bar{\Delta}_k V^{-1}_{k,k'} \Delta_{k'}$$

Stationary point

$$\delta F / \delta \bar{\Delta}_k = -\tanh \frac{\beta E_k}{2} \frac{\Delta_k}{2E_k} - V^{-1}_{k,k'} \Delta_{k'} = 0$$

$$\boxed{\Delta_k = -\sum V_{k,k'} \frac{\Delta_{k'}}{2E_{k'}} \tanh\left(\frac{\beta E_{k'}}{2}\right)}$$

BCS gap Eqn k-dep Δ

If $V_{k,k'} = -g \Rightarrow \Delta_k = \Delta$

If $V_{k,k'} > 0$ for some $k, k' \Rightarrow$ k-dependent gap.

$T = 0$

$$\Delta_k = -\sum U_{k,k'} \frac{\Delta_{k'}}{2E_{k'}}$$

$$\underset{+}{\operatorname{sgn} \Delta_k} = - \underset{+}{\operatorname{sgn}(V_{k,k'})}\ \underset{-}{\operatorname{sgn}(\Delta_{k'})}$$

Regions of phase space linked by <u>repulsive</u> interactions will acquire gap functions of <u>opposite</u> sign $\Rightarrow$ $\boxed{\text{nodes in the Gap fn}}$

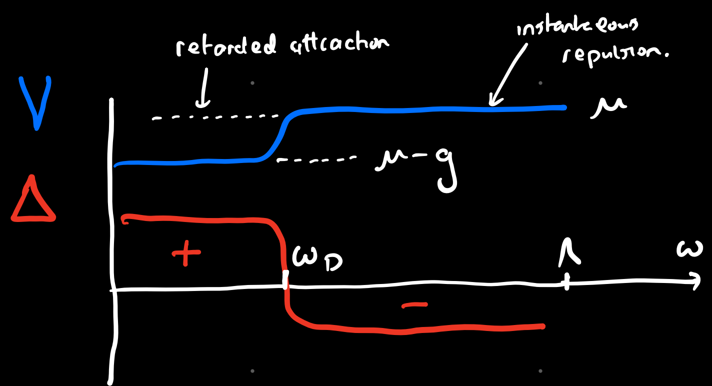
Two important realizations of this effect

- Electron phonon S.c : Interaction is repulsive at high energies. $\Delta_k \to \underset{\text{ISOTROPIC}}{\Delta(\epsilon)}$

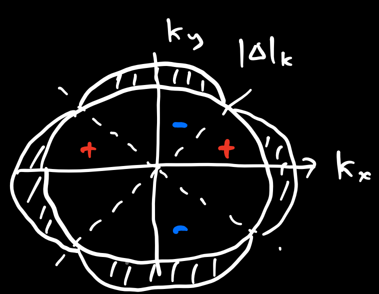
  CHANGES SIGN AS FUNCTION FREQ.

  $V_{k,k'} = \begin{cases} \text{attractive only if } k,k' \sim \epsilon_F \\ \text{repulsive if } k \text{ or } k' \text{ far from Fermi Surface.} \end{cases}$

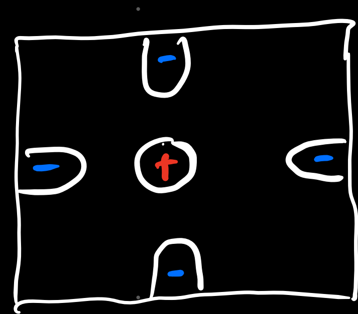
- Anisotropic S.c Gap function has nodes in Momentum space.



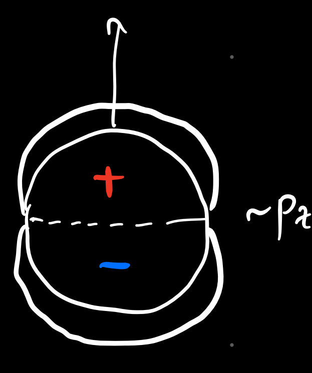
BCS Theory



d-wave s.c



$s^{\pm}$ model for Fe-S.c



"He-3"

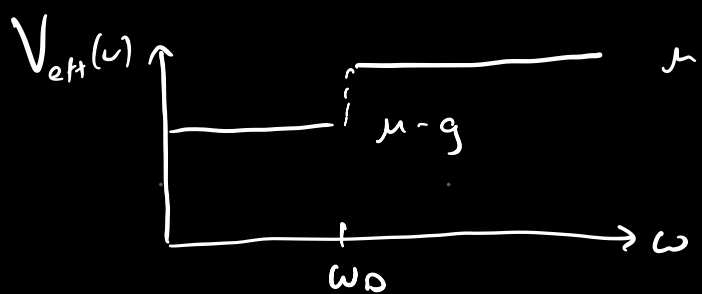
BCS — retardation in interactions

Bardeen-Pines theory.

$$V(\tilde{q},\omega) = \underbrace{\frac{e^2}{\epsilon_0(q^2+k^2)}}_{\text{screened Coulomb}} \Big[\underbrace{1}_{\text{instantaneous}} + \overbrace{\frac{\omega_q^2}{\omega^2-\omega_q^2}}^{\text{Dynamical RETARDED.}} \Big]$$

Anderson-Morel

$$V_{eff}(\omega) = N(0)^{-1} \times \begin{cases} \mu - g & |\omega| < \omega_D \\ \mu & \text{otherwise} \end{cases}$$



$$N(0)\, V_{eff}(t) = N(0) \int V_{eff}(\omega)\, e^{-i\omega t} \frac{d\omega}{2\pi}$$

$$= \mu \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} - g \int_{-\omega_D}^{\omega_D} \frac{d\omega}{2\pi} e^{-i\omega t}$$

$$= \underbrace{\mu\,\delta(t)}_{\text{Instantaneous Repulsion}} \quad - \underbrace{\frac{g\omega_D}{\pi}\left(\frac{\sin\omega_D t}{\omega_D t}\right)}_{\text{Retarded Attraction}}$$

$$\boxed{\mu^* = \frac{\mu}{1+\mu\ln(D/\omega_D)}}$$

RENORMALIZATION OF THE "Coulomb Pseudopotential"

BANDWIDTH

$D/\omega_D \sim \frac{10^5\,K}{500K} \sim 10^2$ $\qquad$ $\ln D/\omega_D \sim 5$ $\qquad$ $\mu^* \sim \frac{\mu}{1+5\mu}$

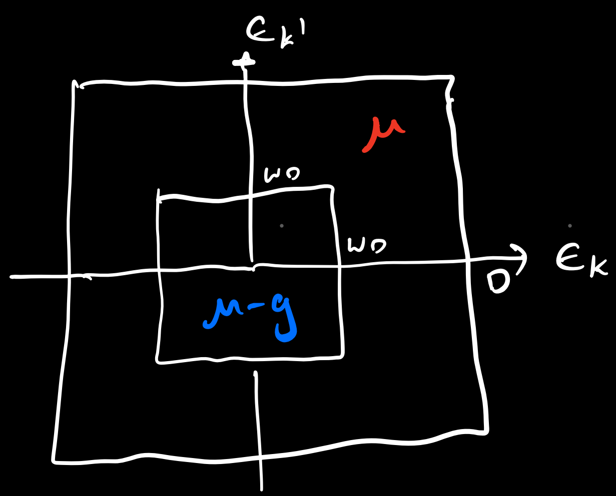
DEBYE ENERGY

$\mu \sim 1$

$\mu^* \sim 1/6$

Choose

$$V_{kk'} = V(\epsilon_k, \epsilon_{k'}) = \frac{1}{N(0)}\begin{cases} \mu - g & |\epsilon_k|, |\epsilon_{k'}| < \omega_D \\ \mu & \text{otherwise} \end{cases}$$



$$\Delta(\epsilon) = -N(0)\int d\epsilon'\, V(\epsilon,\epsilon')\frac{\Delta(\epsilon')}{2E(\epsilon')}$$

$$E = \sqrt{\epsilon^2 + \Delta(\epsilon)^2}$$

Choose

$$\Delta(\epsilon) = \begin{cases} \Delta_1 & |\epsilon| < \omega_D \\ \Delta_2 & D > |\epsilon| > \omega_D . \end{cases}$$

Then

$$\Delta_1 = g - \mu \int_0^{\omega_D} d\epsilon \frac{\Delta_1}{\sqrt{\epsilon^2 + \Delta_1^2}} - \mu \int_{\omega_D}^{D} d\epsilon \frac{\Delta_2}{\sqrt{\epsilon^2 + (\Delta_2)^2}}$$

$$\Delta_2 = -\mu \int_0^{\omega_D} d\epsilon \frac{\Delta_1}{\sqrt{\epsilon^2 + \Delta_1^2}} - \mu \int_0^{D} d\epsilon \frac{\Delta_2}{\sqrt{\epsilon^2 + (\Delta_2)^2}}$$

If $|\Delta_{1,2}| \ll \omega_D, D$ $\qquad \int_0^{\Lambda} dx \frac{\Delta}{\sqrt{x^2 + \Delta^2}} \sim \ln\left(\frac{2\Lambda}{\Delta}\right)$

$$\boxed{\begin{aligned} \Delta_1 &= (g - \mu)\, \Delta_1 \ln\left(\frac{2\omega_D}{\Delta_1}\right) - \mu\, \Delta_2 \ln\left(\frac{D}{\omega_D}\right) \\ \Delta_2 &= -\mu\, \Delta_1 \ln \frac{2\omega_D}{\Delta_1} - \mu\, \Delta_2 \ln \frac{D}{\omega_D} \end{aligned}}$$

$$\left(1+\mu \ln \frac{D}{\omega_D}\right)\Delta_2 = -\mu\,\Delta_1 \ln\left(\frac{2\omega_D}{\Delta_1}\right)$$

$$\Delta_2 = -\mu^* \Delta_1 \ln\left(\frac{2\omega_D}{\Delta_1}\right)$$

$$\boxed{\mu^* = \frac{\mu}{1+\mu \ln \frac{D}{\omega_D}}}$$

$$\Delta_1 = (g-\mu)\Delta_1 \ln\frac{2\omega_D}{\Delta_1} + \Delta_1 \mu\mu^* \ln\frac{2\omega_D}{\Delta_1}\ln\frac{D}{\omega_D}$$

$$= \Delta_1\left[(g-\mu) + \frac{\mu^2 \ln\frac{D}{\omega_D}}{1+\mu\ln D/\omega_D}\right]\ln\frac{2\omega_D}{\Delta_1}$$

$$= \Delta_1\left[g-\mu^*\right]\ln\frac{2\omega_D}{\Delta_1}$$

$$\Delta_1 = 2\omega_D \exp\left[-\frac{1}{(g-\mu^*)}\right]$$

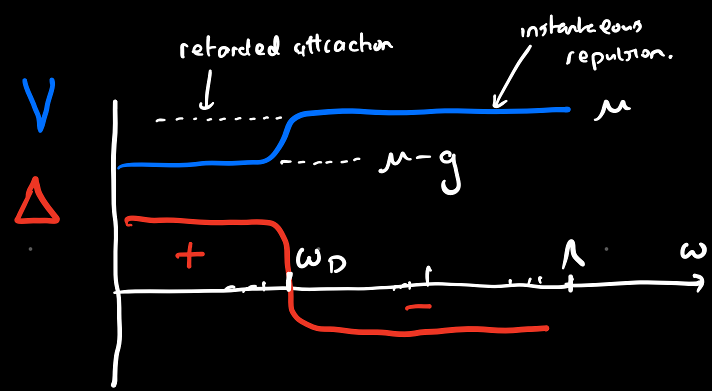
$$\Delta_2 = -\mu^*\Delta_1 \ln\left(\frac{2\omega_D}{\Delta_1}\right)$$

$$= -\frac{\mu^*}{g-\mu^*}\Delta_1$$

- $g - \mu^* > 0$ for S.C. Can still have $(g-\mu) < 0$! i.e an entirely repulsive interaction

In certain metals such alkali + nobel metals, μ^* is still too great for s-wave pairing at ambient pressure.

- $\Delta(\omega)$ changes sign $\sim \omega_D$.



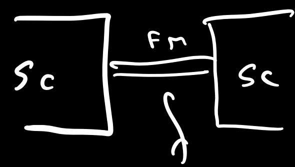
In the time domain, the gap function contains an essentially instantaneous $-$ve component + retarded positive component

$$\Delta(t-t') = -|\Delta_2|\,\delta(t-t') + \Delta_1 \frac{\omega_D}{\pi}\left[\frac{\sin \omega_D(t-t')}{\omega_D(t-t')}\right]$$

instantaneous

retarded

$$\sim \langle \psi_\downarrow(t)\, \psi_\uparrow(t') \rangle$$

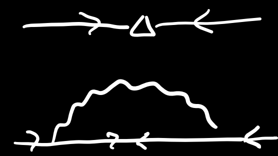


(Berezinski, Eschrig + others)

odd in time

$\Delta(\omega) \propto \omega$!

MIGDAL - ELIASHBERG THEORY



BCS

ELIASHBERG

Simple models.

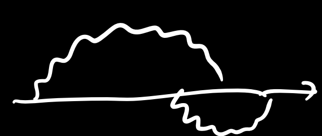
$$H = \sum \omega_q a_q^\dagger a_q + \sum \alpha_{\vec{q}}\, c_{k-q}^\dagger c_k \left(a_q + a_{-q}^\dagger\right)$$

Simpler model "Holstein model" $\omega_q = \omega_0$ constant

Einstein phonon interacting + electrons

QED $\sim \frac{1}{137} \sim \frac{e^2}{\hbar c}$ weak $\therefore$ perturbation theory.

E-P $\frac{\omega_0}{\epsilon_F} \ll 1$

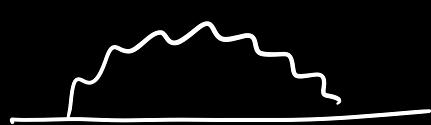


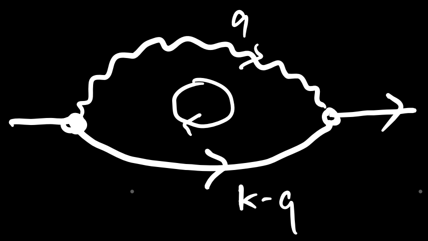
$\sim O\left(\frac{\omega_D}{\epsilon_F}\right)\Sigma_0$

$\ll \Sigma_0$

Neglect
Vertex Corrections

$\Sigma_0 =$

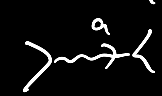




$$= \Sigma(k) = T\sum_{q} (i\alpha_q)^2 \mathcal{G}^0(k-q)\, D(q)$$

$$\Sigma(k) = -T\sum \alpha_q^2\, G(k-q)\, D(q)$$

$$D(q) = D(\vec{q}, i\nu_n) = \frac{2\omega_q}{(i\nu_n)^2 - \omega_q^2} = \left[\frac{1}{i\nu_n - \omega_q} + \frac{1}{-i\nu_n - \omega_{-\vec{q}}}\right]$$



$$\Sigma(\vec{k}, i\omega_n) = -T\sum_{\vec{q}, i\nu_n}\left[\frac{1}{i\nu_n - \omega_q}\,\frac{1}{i\omega_n - i\nu_n - \epsilon_{k-q}} - (\omega_q \to -\omega_q)\right]\alpha_q^2$$

$$-T\sum_{i\nu_n} F(i\nu_n) = \oint \frac{dz}{2\pi i}\, n(z) F(z) \qquad n(\nu) = \frac{1}{e^{\beta\nu}-1}$$

around poles of F

$$f(\epsilon) = \frac{1}{e^{\beta\epsilon}+1}$$

$$\Sigma(\vec{k}, i\omega_n) = \sum \alpha_q^2 \left[\underbrace{\frac{1 + n(\omega_q) - f(\epsilon_{k-q})}{i\omega_n - (\omega_q + \epsilon_{k-q})}}_{\text{(stimulated phonon emission by electron)}} + \underbrace{\frac{n(\omega_q) + f(\epsilon_{k-q})}{i\omega_n - (\epsilon_{k-q} - \omega_q)}}_{\text{thermal phonon absorption by } e^-}\right]$$

Holstein model

$$\Sigma(z) = N(0)\alpha^2 \int d\epsilon \left[\frac{1 + n - f}{z - (\epsilon_k + \omega_0)} + \frac{n + f}{z - (\epsilon_k - \omega_0)}\right]$$

Independent of momentum! — Local in space.

Remarkably, the Kohlstein form works very well for conventional metals, with an appropriate generalization of phonon spectrum $\alpha^2 \longrightarrow \underset{\uparrow \text{ e-p int}}{\alpha^2(\omega)} \underset{\uparrow \text{ phonon density of states.}}{F(\omega)}$

$$\frac{1}{v_F}\nabla_k \Sigma \sim \frac{\Sigma}{\epsilon_F} \qquad \frac{\left(\frac{1}{v_F}\nabla_k \Sigma\right)}{\frac{\partial \Sigma}{\partial \omega}} \sim \frac{\omega_D}{\epsilon_F} \sim 5\%.$$

$$\frac{\partial \Sigma}{\partial \omega} \sim \frac{\Sigma}{\omega_D}$$

$$\Sigma(z) = \frac{\int dS\, \Sigma(\vec{k}, z)}{\int dS}$$

$$\Sigma(\vec{k}, i\nu_n) = \sum_{k'} \alpha^2_{k-k'} \left[\frac{1 + n_{k-k'} - f(\epsilon_{k'})}{i\omega_n - (\omega_{k-k'} + \epsilon_{k'})} + \frac{n_{k-k'} + f(\epsilon_{k'})}{i\omega_n - (\epsilon_{k'} - \omega_{k-k'})} \right]$$

$$\sum_{k'} \rightarrow \int dS'\, dk'_p = \int \frac{dS'\, d\epsilon'}{v_F'}$$

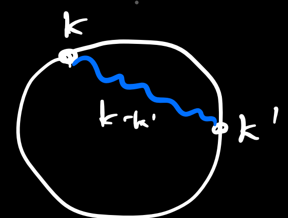
$$\Sigma(z) = \frac{1}{\int ds}\int ds\,\frac{ds'}{v_F}\, d\epsilon'\, \overbrace{(\alpha_{k-k'})^2 \left[\frac{1+n-f}{-} + \frac{n+f}{-}\right]}^{\alpha^2 F}$$

$$= \frac{1}{\int ds}\int d\epsilon'\, d\nu \int \frac{ds\, ds'}{v_F'}\,\alpha_{k-k'}^2\,\delta(\nu-\omega_{k-k'})$$

$$\times \left[\frac{1+n(\nu)-f(\epsilon')}{z-(\nu+\epsilon')} + \frac{n(\nu)+f(\epsilon')}{z-(\epsilon'-\nu)}\right]$$

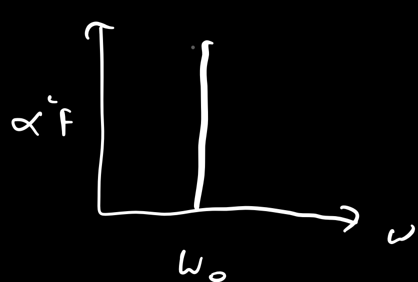
$$\Sigma(z) = \int d\epsilon' \int d\nu\, \alpha^2(\nu)F(\nu)\left[\frac{1+n(\nu)-f(\epsilon')}{z-(\nu+\epsilon')} + \frac{n(\nu)+f(\epsilon')}{z-(\epsilon'-\nu)}\right]$$

$$F(\nu) = \frac{1}{\int ds}\int \frac{ds\, ds'}{v_F'}\,\delta(\nu-\omega_{k-k'})$$



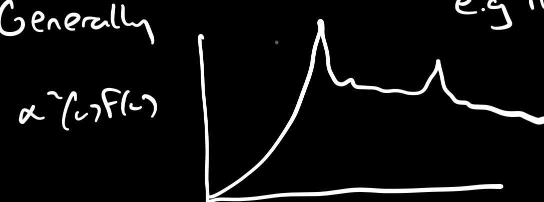
$$\alpha^2(\nu)F(\nu) = \frac{1}{\int ds}\int \frac{ds\, ds'}{v_F'}\,(\alpha_{k-k'})^2\,\delta(\nu-\omega_{k-k'})$$

Holstein



Generally

e.g. H.g.



Next time : use the Nambu propagator

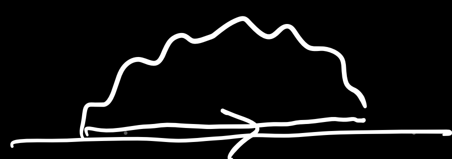
$$\mathcal{G}(\vec{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_k \tau_3 - \underline{\Sigma}(i\omega_n)}$$

$$\underline{\Sigma}(\omega) = \left((1 - Z(\omega))\omega - \Sigma_3 \tau_3 - \varphi(\omega)\tau_1 \right)$$ Assumed form

$$\mathcal{G}(k,\omega) = \frac{1}{Z(\omega)\omega - (\epsilon_k + \Sigma_3(\omega))\tau_3 - \varphi(\omega)\tau_1}$$

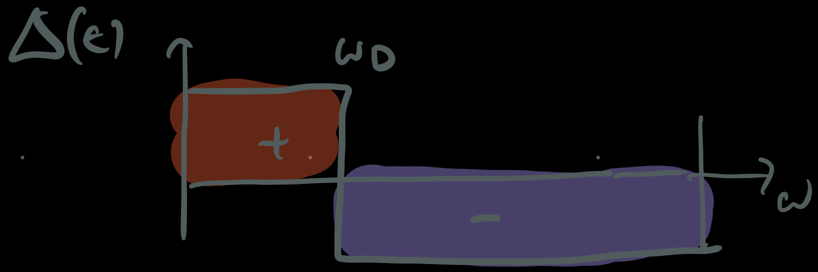
$Z(\omega)$: RENORMALIZATION, QUASIPARTICLE WEIGHT

$\varphi(\omega)$: PAIR

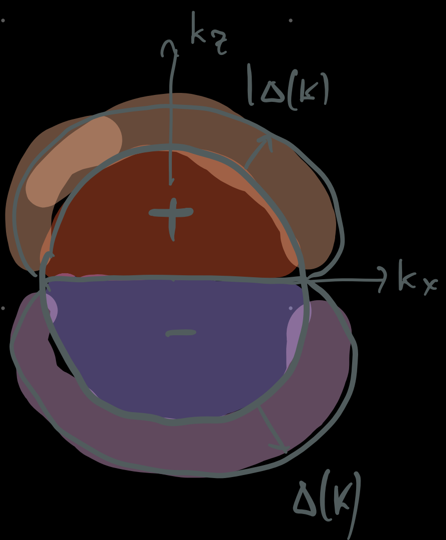
 $= \Sigma(\omega)$

M-E approach.

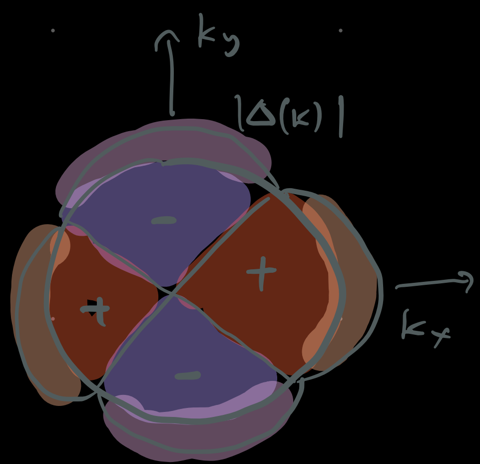
$\boxed{\text{Predecessor of dMFT.}}$



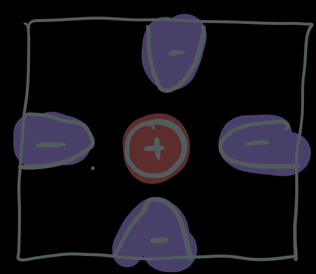
S-wave superconductor



p-wave
(He-3)



d-wave
$YBa_2Cu_3O_7$



$s^{\pm}$ pairing
$Ba_{2-x}K_x Fe_2 As_2$