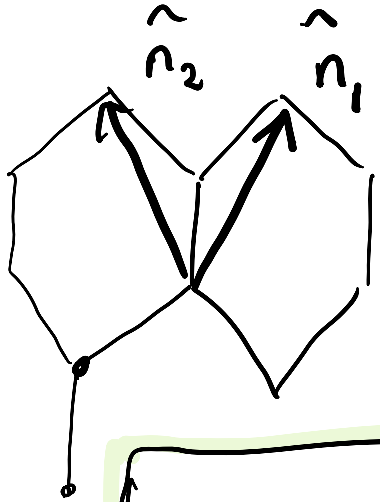
THE TORIC CODE

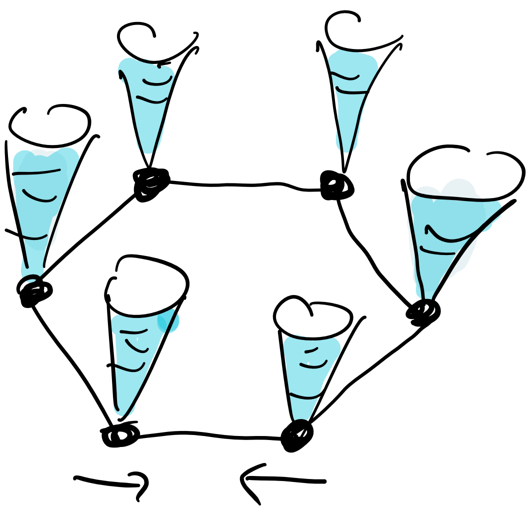
Last time we saw how the Kitaev model can be exactly diagonalized, leading to an exotic ground state with two kinds types of bulk excitation, and, if time reversal is broken, there are also topological edge states.



$$\hat{n}_1 = \left(\frac{\sqrt{3}}{2} a , \frac{3}{2} a \right)$$

$$\hat{n}_2 = \left(-\frac{\sqrt{3}}{2} a , \frac{3}{2} a \right)$$

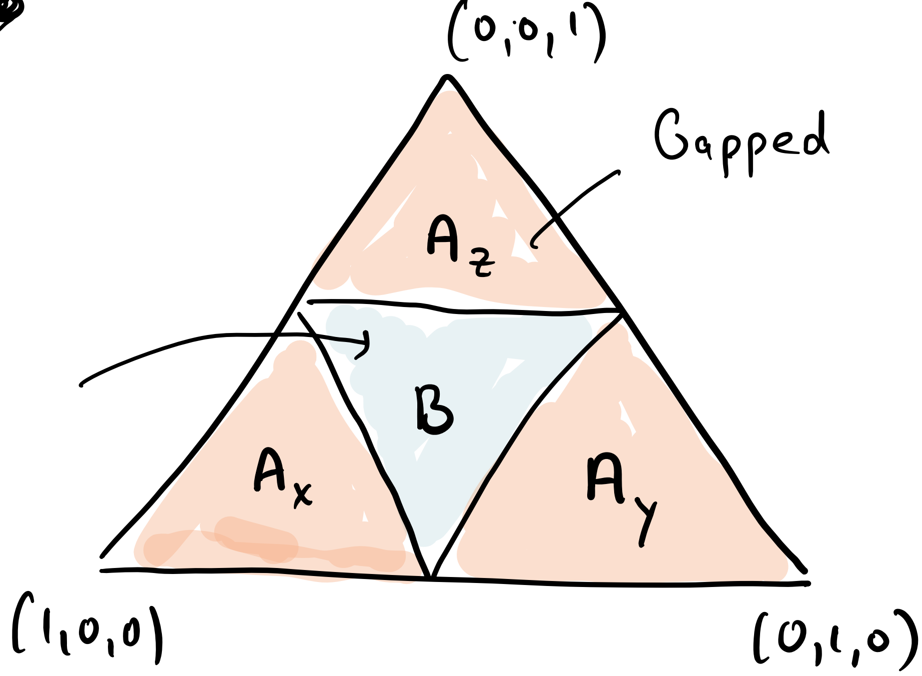
$$\boxed{\begin{aligned} \gamma(k) &= i\left(K_z + K_x e^{i\vec{k}\cdot\vec{n}_1} + K_y e^{i\vec{k}\cdot\vec{n}_2} \right) \\ &= i f_k \\ \epsilon(k) &= \pm \left| \gamma(k) \right| . \end{aligned}}$$



$K_x = K_y = K_z$

$K_z > K_x = K_y$.

Gapless.



FERMION QPS

$$\begin{pmatrix} 0 & if_{\vec{k}} \\ -if^*_{\vec{k}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \mp i e^{-i\theta_k} \end{pmatrix} = \begin{pmatrix} 0 & i|\gamma|e^{i\theta} \\ -i|\gamma|e^{-i\theta} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \mp i e^{-i\theta} \end{pmatrix}$$

$$f_k = |\gamma_k| e^{i\theta_{\vec{k}}} \qquad = \pm |\gamma_k| \begin{pmatrix} 1 \\ \mp i e^{-i\theta_k} \end{pmatrix}$$

$$|\psi_\pm\rangle = |i\rangle\langle i|\psi\rangle \Rightarrow a^\dagger_\pm = c^\dagger_i \langle i|\psi_\pm\rangle$$

$$a^\dagger_\pm(k) = \frac{1}{\sqrt{2}}\left(c^\dagger_{1k} \mp i e^{-i\theta_k} c^\dagger_{2k}\right)$$

$$\hat{\mathcal{H}} = \frac{K}{2} \sum_{k \in \bigcirc} \left[a^\dagger_+(\vec{k}) a_+(\vec{k}) - a^\dagger_-(\vec{k}) a_-(\vec{k}) \right]$$

$$f_k = f^*_{-k} \qquad a^\dagger_-(-k) = \frac{1}{\sqrt{2}}\left(c^\dagger_{1-k} + i e^{+i\theta_k} c^\dagger_{2-k}\right)$$

$$\theta_k = -\theta_{-k}$$

$$= a_+(k)$$

$$\frac{1}{2}\left[a^\dagger_+(k) a_+(k) - a_+(k) a^\dagger_+(k)\right] = \left(a^\dagger_+(k) a_+(k) - \frac{1}{2}\right)$$

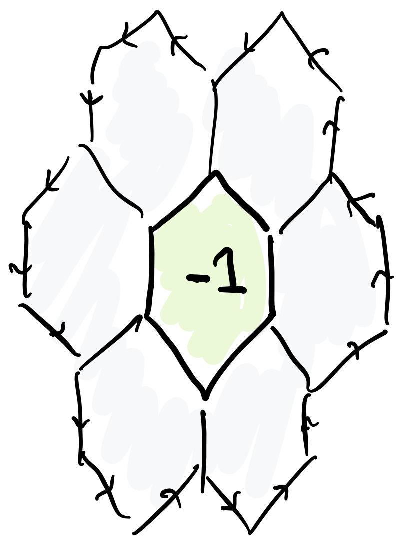
$$\mathcal{H} = \sum_{k \in BZ} |\gamma_k| \left[a_+^\dagger(k) a_+(k) - \frac{1}{2} \right]$$

- Only positive energy excitations : MAJORANA's
- G.S.E $\quad E_g = -\frac{1}{2} \sum_k |\gamma_k|.$

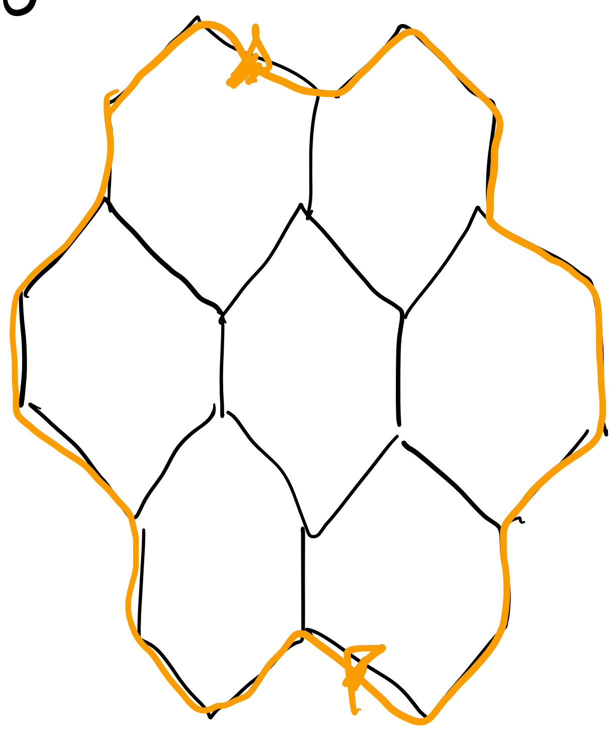
Types of excitation

- Fermions

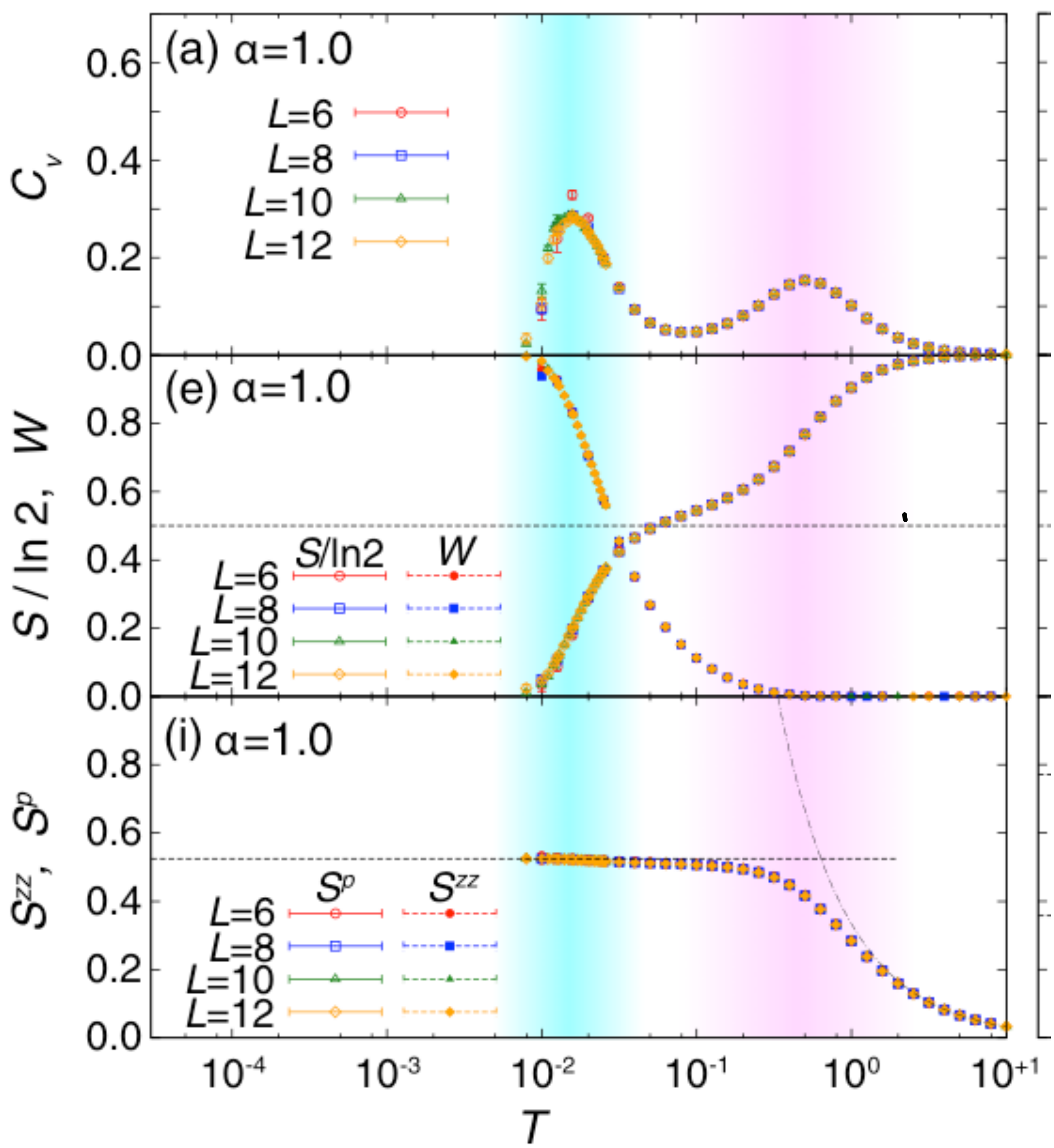
- Vortices.

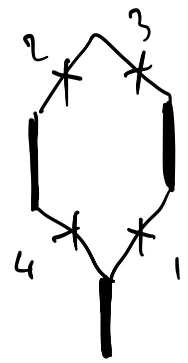


- Edge excitations (broken T.R.)



SPECIFIC HEAT





$$E_V \sim K_{eff} \sim \frac{K_x^2 K_y^2}{16 K_z^3}$$

Factor of $\frac{1}{16} = \frac{8}{4.8.4}$

$$\boxed{W = \langle W_p \rangle,}$$

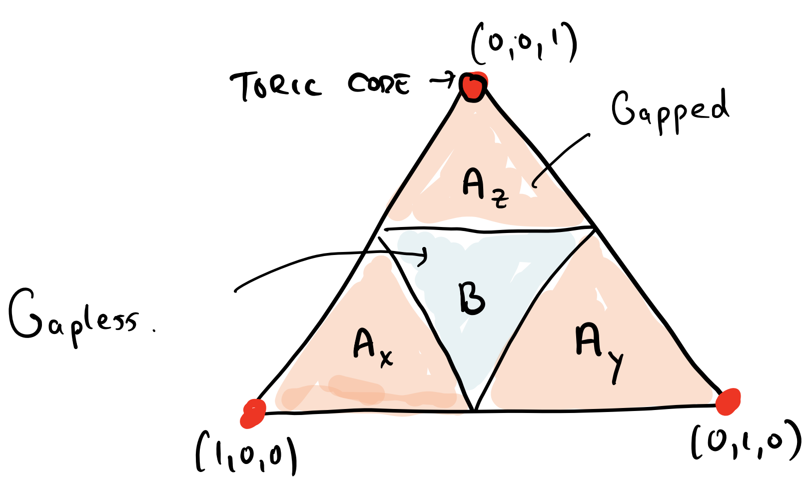
Vortex order

$$\boxed{S^{\ell\ell} = \frac{2}{N} \sum \langle \sigma_j^{\ell} \sigma_k^{\ell} \rangle}$$

ISING ORDER

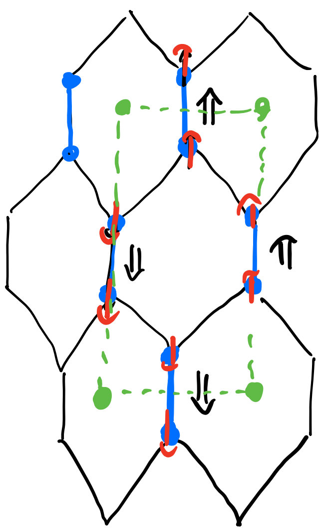
Nasu, Udayana & Motome, PRB 91, 115122 (2015)

Properties of gapped phases



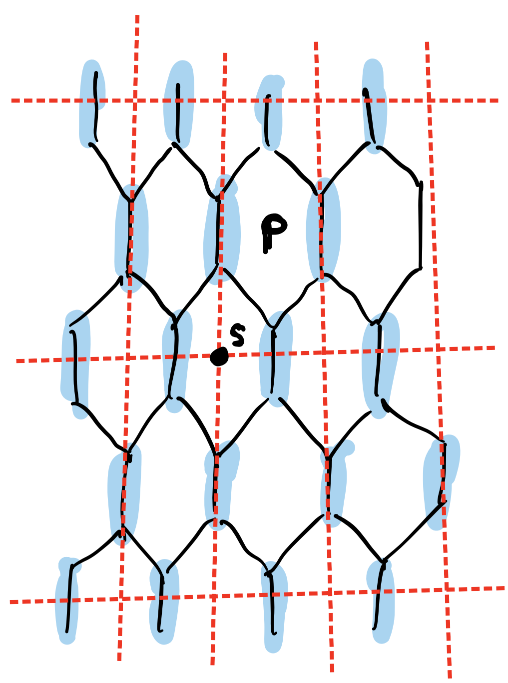
- Around a vortex $\prod u_{ij} = -1$

- Additional insight is gained from the extreme limit $J_x = J_y = 0$, which maps onto the "TORIC CODE"

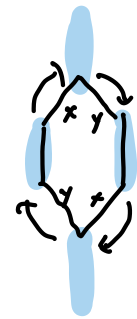


- $$\left.\begin{array}{l} |\Uparrow\rangle = |\uparrow\uparrow\rangle \\ |\Downarrow\rangle = |\downarrow\downarrow\rangle \end{array}\right\} \text{effective spin.}$$

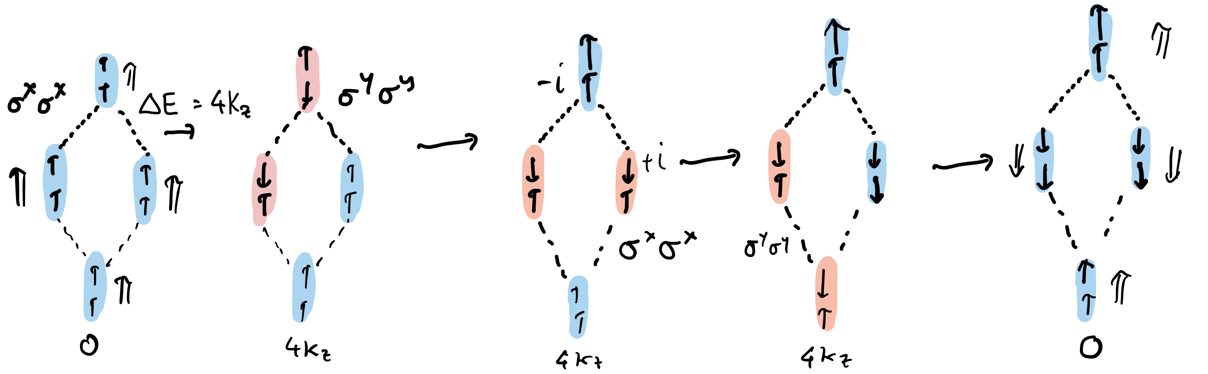
- Lie on bonds of a new lattice.



- Doing perturbation theory on the weak links, acting sequentially around a hexagon $\quad H_{eff}^{(4)} = V G_0 V G_0 V G_0 V$



$$V = \sigma_j^x \sigma_k^x \quad \text{or} \quad \sigma_j^y \sigma_k^y$$

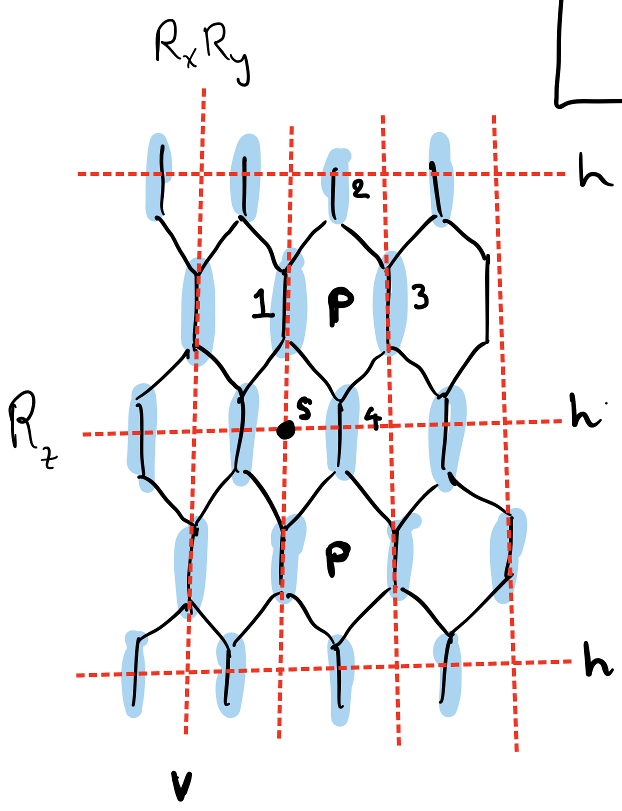


$-\frac{8}{64}$

Gives

$$\frac{8}{64} - \frac{8}{64} + \frac{8}{128}$$

$$\boxed{\begin{gathered} H_{eff}^{(4)} = \text{const} - \frac{K_x^2 K_y^2}{16 K_z^3} \sum_p Q_p \\ Q_p = \sigma_{p_1}^y \sigma_{p_2}^z \sigma_{p_3}^y \sigma_{p_4}^z \end{gathered}}$$



"Plaquet terms" $\quad B_p$

"Star terms" $\quad A_s$

$\left[\begin{array}{lll} \text{Rotate about z} & \left\{\begin{array}{l}\sigma_y \to \sigma_x \\ \sigma_z \to \sigma_z\end{array}\right. & R_z = X \text{ on horizontal links} \\ \text{Rotate about y then x} & \left\{\begin{array}{l}\sigma_y \to \sigma_z \\ \sigma_z \to \sigma_x\end{array}\right. & R_x R_y = Y \text{ on vertical links} \end{array}\right.$

$$\mathcal{H} \longrightarrow U^{\dagger}\mathcal{H}U \qquad U = \underbrace{\sum X_j}_{\text{horizontal}} \ \underbrace{\sum Y_j}_{\text{vertical}}$$

$$X_j = R_z(j) = e^{i\pi/4\,\sigma_z(j)}$$

$$Y_j = e^{i\pi/4\,\sigma_y(j)}\,e^{i\pi/4\,\sigma_x(j)}$$

$$H_T = -J_e \underbrace{\sum_s A_s}_{\text{ELECTRIC}} - J_m \underbrace{\sum_p B_p}_{\text{MAGNETIC}}$$

$A_s = \prod \sigma_j^x$

Star

(FLIP SPINS)

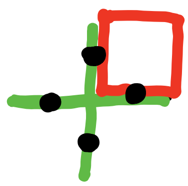
$B_p = \prod \sigma_j^z$

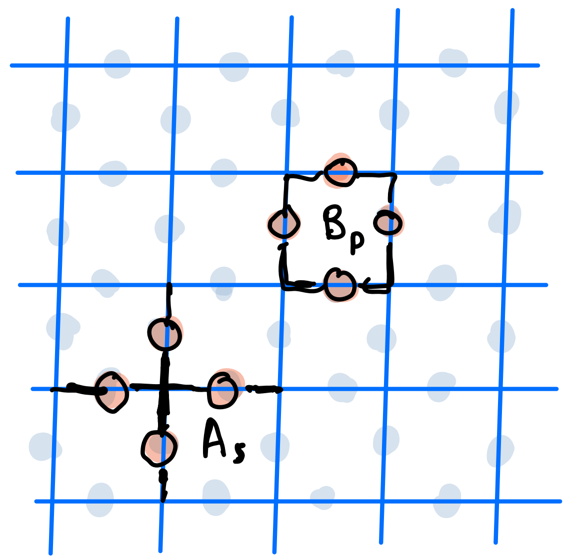
plaquet

TORIC CODE

- A's & B's all commute + each other

(SHARE ZERO OR TWO EDGES

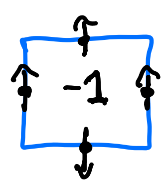




- Vortices correspond to loops with an odd # of spin flips

$$\hat{\sigma}_j^z |s\rangle = s_j |s\rangle$$

$s_j = \pm 1$



$$\boxed{W_p = \prod_{j \in \partial p} s_j}$$

$W_p = -1$ VORTEX.

Minimize energy $\Rightarrow$ $B_p |\psi\rangle = |\psi\rangle$

Requires states with <u>NO</u> <u>VORTICES</u>

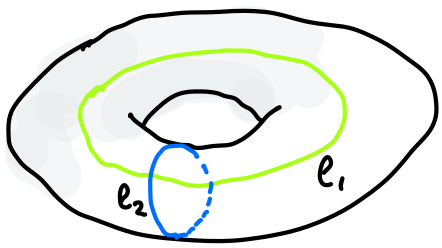
$$\Longrightarrow \quad |\Psi\rangle = \sum_{\{s:\, W_p(s)=1\ \forall p\}} c_s\, |s\rangle$$

The condition $A_s|\Psi\rangle = |\Psi\rangle$ implies the state is unaffected by flipping spins on one star.

$\Longrightarrow$ c_s equal for all configurations with no vortices.

TOPOLOGICAL G.S

$$W_\ell(s) = \prod_{j\in\ell} s_j\,, \quad \ell = \ell_1, \ell_2$$



LARGE CYCLES.

Absence of vortices means the $W_\ell(s)$ are the same for all paths ℓ_1 & all paths ℓ_2.

$W_\ell(s) = \pm 1$

"Cohomology class of vortex free states"

$$|\Psi\rangle = \sum c_{W_{\ell_1} W_{\ell_2}} |s\rangle$$

A_s star operator does not affect $W_\ell(s)$.

$$\boxed{D = (4)^g}$$