MANY BODY PHYSICS: 621. Spring 2022

Exercise 3. (Due Apr 22)

1. (Alternative derivation of the electrical conductivity.)

   In our treatment of the electrical conductivity, we derived

$$\sigma^{ab}(i\nu_n) = e^2 \frac{T}{\nu_n} \sum_{\mathbf{k}, i\omega_r} v^a_{\mathbf{k}} v^b_{\mathbf{k}} \left[G(\mathbf{k}, i\omega_r + i\nu_n) G(\mathbf{k}, i\omega_r) - G(\mathbf{k}, i\omega_r)^2 \right]$$

   This integral was carried out by first integrating over momentum, then integrating over frequency. This techique is hard to generalize and it is often more convenient to integrate the expression in the opposite order. This is the topic of this question. Consider the case where

$$G(\mathbf{k}, i\omega_r) = \frac{1}{i\omega_r - \epsilon_{\mathbf{k}} - \Sigma(i\omega_r)}$$

   and $\Sigma(i\omega_r)$ is any momentum-independent self-energy and $\epsilon_{\mathbf{k}} = (k^2/2m) - \mu$ is a quadratic spectrum.

   (a) By rewriting the momentum integral as an integral over kinetic energy ϵ and, angle show that the conductivity can be rewritten as $\sigma^{ab}(i\nu_n) = \delta^{ab}\sigma(i\nu_n)$, where

$$\sigma(i\omega_n) = \frac{ne^2}{m} \frac{1}{\nu_n} \int_{-\infty}^{\infty} d\epsilon \; T \sum_{i\omega_r} \left[G(\epsilon, i\omega_r + i\nu_n) G(\epsilon, i\omega_r) - G(\epsilon, i\omega_r)^2 \right].$$

   and

$$G(\epsilon, z) \equiv \frac{1}{z - \epsilon - \Sigma(z)}$$

   (b) Carry out the Matsubara sum in the above expression to obtain

$$\sigma(i\omega_n) = \frac{ne^2}{m} \frac{1}{\nu_n} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \int_{-\infty}^{\infty} d\epsilon f(\omega) \left[G(\epsilon, \omega + i\nu_n) + G(\epsilon, \omega - i\nu_n) \right] A(\epsilon, \omega),$$

   where $A(\epsilon, \omega) = ImG(\epsilon, \omega - i\delta)$. (*Hint - replace* $T \sum_n \rightarrow -\int \frac{dz}{2\pi i} f(z)$, *and notice that while* $G(\epsilon, z)$ *has a branch cut along* $z = \omega$ *with discontinuity given by* $G(\epsilon, \omega - i\delta) - G(\epsilon, \omega + i\delta) = 2iA(\epsilon, \omega)$, *while* $G(\epsilon, z + i\nu_n)$ *has a similar branch cut along* $z = \omega - i\nu_n$. *Wrap the contour around these branch cuts and evaluate the result*).

(c) Carry out the energy integral in the above expression to obtain

$$\begin{aligned}\sigma(i\nu_n) &= \frac{ne^2}{m}\frac{1}{\nu_n}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}f(\omega) \\ &\times \left[\frac{1}{i\nu_n-(\Sigma(\omega+i\nu_n)-\Sigma(\omega-i\delta))}-\frac{1}{i\nu_n-(\Sigma(\omega+i\delta)-\Sigma(\omega-i\nu_n))}\right].\end{aligned}$$

(d) Carry out the analytic continuation in the above expression to finally obtain

$$\begin{aligned}\sigma(\nu+i\delta) &= \frac{ne^2}{m}\int_{-\infty}^{\infty}d\omega\left[\frac{f(\omega-\nu/2)-f(\omega+\nu/2)}{\nu}\right]\times \\ &\quad \frac{1}{-i\nu+i(\Sigma(\omega+\nu/2+i\delta)-\Sigma(\omega-\nu/2-i\delta))}. \end{aligned} \tag{1}$$

(e) Show that your expression for the optical conductivity can be rewritten in the form

$$\sigma(\nu+i\delta) = \frac{ne^2}{m}\int_{-\infty}^{\infty}d\omega\left[\frac{f(\omega-\nu/2)-f(\omega+\nu/2)}{\nu}\right]\frac{1}{\tau^{-1}(\omega,\nu)-i\nu Z(\omega,\nu)}. \tag{2}$$

where

$$\tau^{-1}(\omega,\nu) = \mathrm{Im}\left[\Sigma(\omega-nu/2-i\delta)+\Sigma(\omega+\nu/2-i\delta)\right] \tag{3}$$

is the average of the scattering rate at frequencies $\omega \pm \nu/2$ and

$$Z^{-1}(\omega,\nu)-1 = -\frac{1}{\nu}\mathrm{Re}\left[\Sigma(\omega-\nu/2)-\Sigma(\omega+\nu/2)\right]$$

is a kind of "wavefunction renormalization".

(f) Show that if the ω dependence of Z and τ^{-1} can be neglected, one arrives at the phenomenological form

$$\sigma(\nu) = \frac{ne^2}{m}\left[\frac{1}{\tau^{-1}(\nu)-i\nu Z^{-1}(\nu)}\right]$$

This form is often used to analyze optical spectra.

(g) Show that the zero temperature conductivity is given by the thermal average

$$\sigma(\nu+i\delta) = \frac{ne^2\tau}{m} \tag{4}$$

where $\tau^{-1} = 2Im\Sigma(0-i\delta)$.

2. Generalize the BCS solution to the case where the gap has a finite phase $\Delta = |\Delta| e^{i\phi}$. Show that in this case, the eigenvectors of the BCS mean-field hamiltonian are

$$\begin{aligned} u_{\mathbf{k}} &= e^{i\phi/2}\left(\frac{1}{2}+\frac{\epsilon_{\mathbf{k}}}{2E_{\mathbf{k}}}\right)^{\frac{1}{2}} \\ v_{\mathbf{k}} &= e^{-i\phi/2}\left(\frac{1}{2}-\frac{\epsilon_{\mathbf{k}}}{2E_{\mathbf{k}}}\right)^{\frac{1}{2}} \end{aligned} \tag{5}$$

while the BCS ground-state is given by

$$|BCS(\phi)\rangle = \prod_{\mathbf{k}} (u^*_{\mathbf{k}} + v^*_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} c^\dagger_{\mathbf{k}\uparrow})|0\rangle. \tag{6}$$

3. Explicit calculation of the Free energy.

   (a) Assuming that the Debye frequency is a small fraction of the band-width, show that the difference between the superconducting and normal state Free energy can be written as the integral

   $$\mathcal{F}_S - \mathcal{F}_N = -2TN(0)\int_{-\omega_D}^{\omega_D} d\epsilon \ln\left[\frac{\cosh\left(\frac{\sqrt{\epsilon^2+|\Delta|^2}}{2T}\right)}{\cosh\left(\frac{\epsilon}{2T}\right)}\right] + V\frac{|\Delta|^2}{g_0}.$$

   Why is this free energy invariant under changes in the phase of the gap parameter $\Delta \to \Delta e^{i\phi}$?

   (b) By differentiating the above expression with respect to Δ, confirm the zero temperature gap equation,

   $$\frac{V}{gN(0)} = \int_0^{\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2+\Delta_0^2}},$$

   where $\Delta_0 = \Delta(T=0) =$ is the zero temperature gap and use this result to eliminate g_0, to show that the free energy can be written

   $$\mathcal{F}_S - \mathcal{F}_N = N(0)\Delta_0^2\, \Phi\left[\frac{\Delta}{\Delta_0}, \frac{T}{\Delta_0}\right]$$

   where the dimensionless function

   $$\Phi(\delta, t) = \int_0^\infty dx \left\{ -4t \ln\left[\frac{\cosh\left(\frac{\sqrt{x^2+\delta^2}}{2t}\right)}{\cosh\left(\frac{x}{2t}\right)}\right] + \frac{\delta^2}{\sqrt{x^2+1}}\right\}.$$

   Here, the limit of integration have been moved to infinity. Why can we do this without loss of accuracy?

(c) Use Mathematica or Maple to plot the Free energy obtained from the above result, confirming that the minimum is at $\Delta/\Delta_0 = 1$ and the transition occurs at $T_c = 2\Delta_0/3.53$.

4. (a) The superfluid stiffness of a BCS s-wave superconductor is given by

$$Q(T) = \frac{1}{\mu_0 \lambda_L^2(T)} Q_0 \left[1 - 2 \int_{\Delta(T)}^{\infty} d\omega \left(-\frac{df(\omega)}{d\omega} \right) \left(\frac{\omega}{\omega^2 - \Delta^2} \right) \right]$$

where $\lambda_L(T)$ is the London penetration depth at temperature T. Suppose $\Delta(\phi) = \Delta \cos(2\phi)$ is the angular dependence of the gap in a d-wave superconductor. How would this expression change, and how would the temperature dependence of the London penetration depth be modified?