Exercise 1. Path Integrals. (Due Fri 17 Feb.)

1. In this problem consider $\hbar = 1$. Suppose $|0\rangle$ is the ground-state of a harmonic oscillator problem, where $b|0\rangle = 0$. Consider the state formed by simultaneously translating this state in momentum and position space as follows:

$$|p, x\rangle = \exp\left[-i(x\hat{p} - p\hat{x})\right]|0\rangle.$$  

By rewriting $\hat{b} = (\hat{x} + i\hat{p})/\sqrt{2}$, $z = (x + ip)/\sqrt{2}$, show that this state can be rewriting as

$$|p, x\rangle = e^{b^\dagger z - z\bar{b}}|0\rangle$$

and show that the coherent state $|z\rangle$ represents a minimum uncertainty wavepacket centered at $(q, p)$ in phase space.

2. (a) Suppose $H = \epsilon c^\dagger c$ the Hamiltonian for an energy level $\epsilon$ that can be occupied by a single fermion. Consider the approximation to the partition function obtained by first dividing up the period $\tau \in [0, \beta]$ into $N$ equal time-slices, $Z_N = \text{Tr}(e^{-\Delta t H})^N$, which is given by

$$Z_N = \int \prod_{j=1}^N \text{d}\bar{c}_j \text{d}c_j \exp\{-S_N\}$$

$$S_N = \sum_{j=1}^N \Delta \tau \left[\bar{c}_j(c_j - c_{j-1})/\Delta \tau + \epsilon \bar{c}_j c_{j-1}\right].$$

(a) Show that $Z_3$ can be written as a “toy functional integral”,

$$Z_3 = \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \exp\left\{-\left(\begin{array}{c} \bar{c}_3 \\ \bar{c}_2 \\ \bar{c}_1 \end{array}\right) \left(\begin{array}{ccc} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1 \end{array}\right) \left(\begin{array}{c} c_3 \\ c_2 \\ c_1 \end{array}\right)\right\},$$
where $\alpha = 1 - \Delta \tau \epsilon$. In this formula, the discrete time-line is labelled as follows,

$$
\begin{array}{cccccc}
\bar{c}_1 & c_1 & c_2 & c_1 & c_0 = -c_3 \\
\beta = \tau_3 & \tau_2 & \tau_1 & 0
\end{array}
$$

(4)

where $(\bar{c}_j, c_j)$ are the conjugate Grassman variables at each discrete time $\tau_j = j \Delta \tau$.

(b) Evaluate $Z_3$.

(c) Generalize the result to $N$ time slices and obtain an expression for $Z_N$. What is the limiting value of your result as $N \to \infty$?

(d) Repeat the calculation for a boson $H = \epsilon \hat{b}^\dagger \hat{b}$

3. A system of weakly interacting superfluid bosons is described by the action

$$
S = \int_0^\beta d\tau d^3x \left[ \bar{\psi} \left( \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right) \psi + \frac{g}{2} |\psi|^4 \right].
$$

(5)

(a) Ignoring fluctuations of the Bose Field, sketch the Free energy ($F = S/\beta$) as a function of the magnitude of a uniform Bose field, for $\mu > 0$ $\mu = 0$ and $\mu < 0$.

(b) Assuming you can ignore fluctuations of the Bose field, when $\mu > 0$, what is the uniform equilibrium value of the Bose field $\psi(x)$?

(c) What is the corresponding coherent state?

(d) Assume that the superfluid lives on a torus. Suppose the phase of the wavefunction winds through $l$ turns as one goes around the torus in the $x$ direction. Write down the wavefunction for this state. What is the superfluid velocity of this state and why is the superflow persistent?

(e) (Harder.) In class we considered the fluctuations in a “cartesian” basis, expanding the action to Gaussian order in terms of $\delta \psi(x)$ and $\delta \bar{\psi}(x)$. Carry out the calculation in the “radial” basis - i.e write the action in terms $\psi(x) = r(x)e^{i\phi(x)}$ and expand it to Gaussian order in $\delta r(x)$ and $\delta \phi(x)$. (Hint, choose $\phi_0 = 0$, so that $\delta \psi = \delta r(x) + ir_0 \delta \phi(x) + O(\delta \psi^2)$. Note also that total derivatives, like $\int_0^\beta r \partial_\tau r = \frac{1}{2} \int_0^\beta \partial_\tau (r^2) = 0$ vanish). Confirm that you obtain the same results for the low-lying excitation spectrum as we obtained in class.