MANY BODY PHYSICS: 621. Spring 2022

Solutions to Problems 1 . February 28th, 2022.

1. (a) This question was an exercise to help you visualize the meaning of a coherent state as a Gaussian wave packet centered around a shifted momentum and position. The ground-state of a harmonic oscillator is a wavepacket centered at $x = 0$ and $p = 0$. Consider the state

$$|p, x\rangle = \exp\left[-i(x\hat{p} - p\hat{x})\right]|0\rangle. \tag{1}$$

By rewriting $z = (x + ip)/\sqrt{2}$, $b = (\hat{x} + i\hat{p})/\sqrt{2}$ it follows that

$$b^\dagger z - \bar{z}b = \frac{1}{2}\left[(\hat{x} - i\hat{p})(x + ip) - (\hat{x} + i\hat{p})(x - ip)\right] = i(p\hat{x} - x\hat{p}) \tag{2}$$

so that $|p, x\rangle = \exp\left[b^\dagger z - \bar{z}b\right]|0\rangle$. Next, using the fact that $e^{A+B} = e^A e^B e^{-\frac{1}{2}[\hat{A},\hat{B}]}$, (provided $[A, B]$ commutes with A and B). I apologize, as the minus sign before the commutator was missing in the question. Putting $\hat{A} = b^\dagger z$ and $\hat{B} = -\bar{z}b$, so that $[A, B] = -\bar{z}z[b^\dagger, b] = \bar{z}z$ and thus

$$|p, x\rangle = e^{-\bar{z}z/2}e^{\hat{b}^\dagger z}e^{-\bar{z}\hat{b}}|0\rangle = e^{-\bar{z}z/2}e^{\hat{b}^\dagger z}|0\rangle = e^{-\bar{z}z/2}|z\rangle. \tag{3}$$

In other words, the coherent state $|z\rangle$, is, up to a normalization factor, the translation of the vacuum to a state that is centered around $z = (x + ip)/\sqrt{2}$. Since a translation preserves the shape of the ground-state wavefunction, the state remains a minimum uncertainty wavepacket, with $\Delta p^2 = \Delta x^2 = 1/2$.

2. In this question, I wanted to check that you had understood how to set up a fermionic path integral.

   (a) The first step in setting up the path integral is to write the Trace using coherent states. Let us define

$$|c\rangle = \exp[\hat{c}^\dagger c]|0\rangle, \qquad \langle\bar{c}| = \langle 0|\exp[\bar{c}\hat{c}].$$

Now, for a general operator $\hat{A}$, the matrix element between these coherent states is

$$\langle\bar{c}|\hat{A}|c\rangle \;=\; A_{oo}\langle\bar{c}|0\rangle\langle 0|c\rangle + A_{01}\langle\bar{c}|0\rangle\langle 1|c\rangle + A_{10}\langle\bar{c}|1\rangle\langle 0|c\rangle + A_{11}\langle\bar{c}|1\rangle\langle 1|c\rangle$$

$$= A_{oo} + A_{01}c + A_{10} + A_{11}\bar{c}c \tag{4}$$

so that the trace may be written

$$Tr[A] = A_{00} + A_{11} = -\int d\bar{c}dce^{\bar{c}c}\langle\bar{c}|\hat{A}|c\rangle \tag{5}$$

Applying this to the partition function,

$$\begin{aligned} Tr[e^{-\beta H}] &= -\int d\bar{c}_3 dc_0 e^{\bar{c}_3 c_0}\langle\bar{c}_3|e^{-\beta H}|c_0\rangle \\ &= \int d\bar{c}_3 dc_3 e^{-\bar{c}_3 c_3}\langle\bar{c}_3|e^{-\beta H}|c_0\rangle, \end{aligned} \tag{6}$$

where we have used the definition, $c_3 = -c_0$. We now use the completeness relation

$$1 = \int d\bar{c}dce^{-\bar{c}c}|c\rangle\langle\bar{c}| \tag{7}$$

to introduce two time-slices into the matrix element, writing $e^{-\beta H} = (e^{-\Delta\tau H})^3 = e^{-\Delta\tau H}\underline{1}_2 e^{-\Delta\tau H}\underline{1}_1 e^{-\Delta\tau H}$, where $\Delta\tau = \beta/3$, so that

$$\langle\bar{c}_3|e^{-\beta H}|c_0\rangle = \int d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \langle\bar{c}_3|e^{-\Delta\tau H}|c_2\rangle\langle\bar{c}_2|e^{-\Delta\tau H}|c_1\rangle\langle\bar{c}_1|e^{-\Delta\tau H}|c_0\rangle e^{-(\bar{c}_2 c_2 + \bar{c}_1 c_1)} \tag{8}$$

so that

$$Tr[e^{-\beta H}] = \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \langle\bar{c}_3|e^{-\Delta\tau H}|c_2\rangle\langle\bar{c}_2|e^{-\Delta\tau H}|c_1\rangle\langle\bar{c}_1|e^{-\Delta\tau H}|c_0\rangle e^{-(\bar{c}_3 c_3 + \bar{c}_2 c_2 + \bar{c}_1 c_1)}.$$

Finally, using the expansion of the matrix element in terms of coherent states,

$$\langle\bar{c}_j|e^{-\Delta\tau H}|c_{j-1}\rangle = e^{\alpha\bar{c}_j c_{j-1}} + O(\Delta\tau^2) \tag{9}$$

where $\alpha = (1 - \Delta\tau\epsilon)$, we obtain with accuracy $O(3\Delta\tau^2)$

$$\begin{aligned} Z_3 &= \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 e^{\alpha[\bar{c}_3 c_2 + \bar{c}_2 c_1 + \bar{c}_1 c_0] - [\bar{c}_3 c_3 + \bar{c}_2 c_2 + \bar{c}_1 c_1]} \\ &= \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \exp\left\{ -(\bar{c}_3, \bar{c}_2, \bar{c}_1) \begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 \\ c_2 \\ c_1 \end{pmatrix} \right\}, \end{aligned} \tag{10}$$

(b) Since this integral is Gaussian, the integral is given by the determinant of the matrix:

$$Z_3 = \det \begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1 \end{pmatrix} = 1 + \alpha^3 \tag{11}$$

(c) Generalizing this result to N time-slices, we obtain

$$\begin{aligned} Z_N &= \det[\mathcal{M}] \\ \mathcal{M} &= \begin{pmatrix} 1 & -\alpha & 0 & \dots & 0 \\ 0 & 1 & -\alpha & \dots & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & -\alpha \\ \alpha & \dots & \dots & \dots & 1 \end{pmatrix} \\ \det[\mathcal{M}] &= 1 + \alpha^N \qquad \text{(by inspection)} \end{aligned} \tag{12}$$

In the limit $N \to \infty$,

$$\alpha^N = \left(1 - \frac{\beta\epsilon}{N}\right)^N \to e^{-\beta\epsilon} \tag{13}$$

so that

$$Z_N \to 1 + e^{-\beta\epsilon}. \tag{14}$$

(d) Repeating the calculation using bosonic coherent states

$$|b\rangle = \exp[\hat{b}^\dagger b]|0\rangle, \qquad \langle\bar{b}| = \langle 0|\exp[\bar{b}\hat{b}].$$

The trace formula is now

$$Tr[A] = \sum_n \langle n|A|n\rangle = \int \frac{d\bar{b}db}{2\pi i} e^{-\bar{b}b}\langle\bar{b}|\hat{A}|b\rangle, \tag{15}$$

while the completeness relation is

$$1 = \int d\bar{b}db e^{-\bar{b}b}|b\rangle\langle\bar{b}|, \tag{16}$$

enabling us to write

$$Z_3 = \int \frac{d\bar{b}_3 db_0}{2\pi i} \prod_{j=1}^{2} \frac{d\bar{b}db}{2\pi i} e^{\alpha[\bar{b}_3 b_2 + \bar{b}_2 b_1 + \bar{b}_1 b_0] - [\overbrace{\bar{b}_3 b_0}^{\bar{b}_3\bar{b}_3} + \bar{b}_2 b_2 + \bar{b}_1 b_1]}$$

$$= \int \prod_{j=1}^{3} \frac{d\bar{b}db}{2\pi i} \exp\left\{ -(\bar{b}_3, \bar{b}_2, \bar{b}_1) \begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ -\alpha & 0 & 1 \end{pmatrix} \begin{pmatrix} b_3 \\ b_2 \\ b_1 \end{pmatrix} \right\}$$
$$= \frac{1}{(1-\alpha^3)}. \tag{17}$$

where we have defined $b_3 = b_0$ in the second step. The same limiting procedure then leads to the result

$$\lim_{N\to\infty} Z_N = \frac{1}{1-e^{-\beta\epsilon}}. \tag{18}$$

3. We are evaluating the partition function

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] \tag{19}$$

where

$$H = -\mu_B B \sigma f^\dagger_\sigma f_\sigma, \tag{20}$$

with an implied summation over the repeated $\sigma = \pm 1$ index. Casting this as a path integral,

$$Z = \int \mathcal{D}[\bar{f}, f] e^{-S},$$

where

$$S = S = \sum_{\sigma=\pm 1} \int d\tau \bar{f}_\sigma \Big(\partial_\tau - \mu_B \sigma B \Big) f_\sigma, \tag{21}$$

we can write

$$Z = \det[\partial_\tau - \mu_B \sigma_z B] = \det[\partial_\tau - \mu_B B]\det[\partial_\tau + \mu_B B] \tag{22}$$

Switching to Fourier space,

$$Z = \prod_{\sigma=\pm} \prod_{i\nu_n} [-i\nu_n - \sigma\mu_B B]$$

The Free energy is then

$$F = -T \sum_{\sigma=\pm 1} \sum_{i\nu_n} \ln[-i\nu_n - \sigma\mu_B B] \tag{23}$$

We can carry out this summation using the contour integral method, rewriting

$$F = \sum_{\sigma=\pm 1} \oint \frac{dz}{2\pi i} f(z) \ln[-z - \sigma\mu_B B], \tag{24}$$

where the integral is anticlockwise around the real axis. The function $\ln[-z+\epsilon]$ has a branch cut along the real axis from $z=\epsilon$ to $z=\infty$, and since $F[\omega+i\delta] - F\omega - i\delta = 2i\mathrm{Im}F(\omega+i\delta)$, it follows that

$$\begin{aligned} F &= \sum_\sigma \int \frac{d\omega}{\pi} f(\omega) \mathrm{Im} \ln[-(\omega+i\delta) - \sigma\mu_B B] \\ &= -\sum_\sigma \int_{-\sigma\mu_B B}^{\infty} d\omega f(\omega) = \left[T\mathrm{n}(1+e^{-\beta x})\right]_{-\sigma\mu_B B}^{\infty} \\ &= \sum_\sigma (-T)\ln[1+e^{\beta\sigma\mu_B B}] \\ &= -T\ln(2+2\cosh[\beta\mu_B B]) \end{aligned} \tag{25}$$

The correct answer is

$$F = -T\ln(2\cosh[\beta\mu_B B])$$

Our answer is correct, given our original Hamiltonian, but it is not correct for a single spin-1/2. We have obtained

$$Z = 2 + Z_{S=1/2} = 2 + 2\cosh[\beta\mu_B B]$$

Had we removed the empty and doubly occupied states, then the partition function becomes

$$Z = 2\cosh[\beta\mu_B B]$$

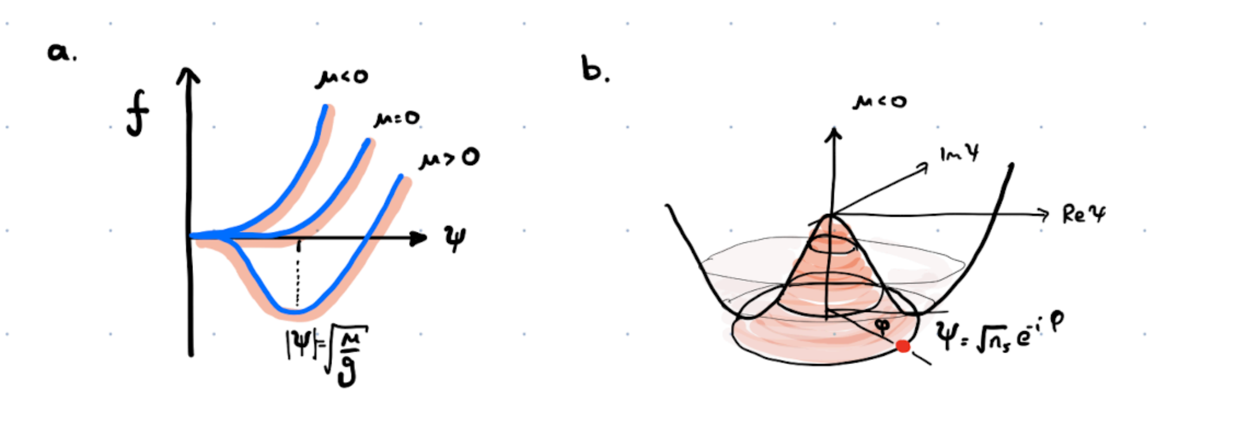

FIG. 1: a) $f[\psi]$ as a function of μ in profile and b) in three dimensions, showing the Mexican hat potential.

4. This problem was an exercise in thinking about the Ginzburg Landau theory for a superfluid, using the concept of a coherent state.

(a) The free energy $f = S/(\beta V)$ per unit volume for a uniform Bose field ψ is given by

$$f = -\mu|\psi|^2 + \frac{g}{2}|\psi|^4 \tag{26}$$

A sketch of the free energy is given in Fig (1).

(b) The minimum of the free energy is determined by $\partial f/\partial|\psi|^2 = -\mu + g|\psi|^2$, i.e $\psi = \sqrt{\frac{\mu}{g}}e^{i\phi}$ where ϕ is the phase of the order parameter and $n_s = \frac{\mu}{g}$ is the superfluid density.

(c) The coherent state for the ground-state is

$$|\psi\rangle = \exp\left[\sqrt{N_s}b^\dagger_{q=0}\right]|0\rangle = \exp\left[\int d^3x\sqrt{n_s}e^{i\phi}\psi^\dagger(x)\right]|0\rangle$$

where $b^\dagger_{\mathbf{q}} = \frac{1}{\sqrt{V}}\int d^3x\psi^\dagger(x)e^{i\mathbf{q}\cdot\mathbf{x}}$ and $N_s = n_sV$ is the number of bosons in the condensate. You can normalize this state by noting that $\langle\psi|\psi\rangle = e^{N_s}$.

(d) In this situation, $\phi(x) = l(2\pi/L)x = \mathbf{Q}\cdot\mathbf{x}$ where L is the circumference of the torus in the x-direction and $\mathbf{Q} = (2\pi l/L, 0, 0)$ so that now

$$|\psi\rangle = \exp\left[\int d^3x\sqrt{n_s}e^{i\mathbf{Q}\cdot\mathbf{x}}\psi^\dagger(x)\right]|0\rangle.$$

Two of you noted that if one reminimizes the action, $\mu \to \tilde{\mu} = \mu - \frac{\hbar^2Q^2}{2m}$, so that now $n_s = \tilde{\mu}/g$. The superfluid velocity is given by

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla\phi = \frac{\hbar}{m}\mathbf{Q} \tag{27}$$

(e) So long as the superfluid velocity is lower than the critical velocity, it costs energy for the walls to create quasiparticles, and the superflow can not decay without a passage of vortices through the medium, which costs a large amount of energy or action, causing the rate of current decay to be exponentially small.

(f) One can include fluctuations into the path integral, writing

$$\psi(x,\tau) = \psi_0 + \frac{1}{\sqrt{\beta V}}\sum_{\mathbf{q},\mu_n} b_{\mathbf{q},n}e^{i(\mathbf{q}\cdot x - \nu_n\tau)}$$

and one can then carry out the Gaussian integration over the fluctuations. The resulting quasiparticle dispersion is given by $\omega_{\mathbf{q}} = \sqrt{(E_{\mathbf{q}} + gn_c)^2 - (gn_c)^2}$, and the correction to the mean-field Free energy derived from these gaussian fluctuations is given by $F = F_o + \frac{T}{2}\mathrm{Tr}\ln[-G^{-1}]$. This correction is approximately given by

$$F = F_0 + T\sum_{\mathbf{q}} \ln[1 - e^{-\beta\omega_{\mathbf{q}}}] \tag{28}$$

where F_0 is the ground-state energy.

5. (a) In this problem, it actually was not necessary to choose the dimension of A to be even. Lets take it to be dimension N. We shall use a coherent state representation to write

$$T = \mathrm{Tr}[\mathcal{M}\mathcal{M}^\dagger] = (-1)^N \int \prod_{i=1,2N} d\bar{c}_i dc_i e^{\bar{c}_i c_i} \langle \bar{c}|\mathcal{M}\mathcal{M}^\dagger|c\rangle \tag{29}$$

Now since $\mathcal{M} = \exp[\frac{1}{2}A_{ij}c_i^\dagger c_j]$ is already normal ordered, it follows that

$$\langle \bar{c}|\mathcal{M}\mathcal{M}^\dagger|c\rangle = \exp\left[\bar{c}_j c_j + \frac{1}{2}A_{ij}\bar{c}_i\bar{c}_j + c_j c_i \bar{A}_{ij}\right] \tag{30}$$

so we can write the trace as

$$\begin{aligned} T = \mathrm{Tr}[\mathcal{M}\mathcal{M}^\dagger] &= (-1)^N \int \prod_{i=1,2N} d\bar{c}_i dc_i \exp\left[2\bar{c}_j c_j + \frac{1}{2}A_{ij}\bar{c}_i\bar{c}_j + c_j c_i \bar{A}_{ij}\right] \\ &= (-1)^N \int \prod_{i=1,2N} d\bar{c}_i dc_i \exp\left[\frac{1}{2}\bar{\psi}\cdot\Lambda\cdot\psi\right] \end{aligned} \tag{31}$$

where the matrix

$$\Lambda = \left(\begin{array}{ccc|ccc} 2 & & & \ddots & & \\ & \ddots & & & \underline{A}_{ij} & \\ & & 2 & & & \ddots \\ \hline \ddots & & & -2 & & \\ & \underline{A}^\dagger_{ij} & & & \ddots & \\ & & \ddots & & & -2 \end{array}\right) \tag{32}$$

and

$$\psi = \begin{pmatrix} c_1 \\ \vdots \\ c_N \\ \bar{c}_1 \\ \vdots \\ \bar{c}_N \end{pmatrix} \tag{33}$$

It follows that

$$T = (-1)^N \sqrt{\det[-\Lambda]} = \sqrt{\det[\Lambda]}, \tag{34}$$

where we have been loose with the minus signs. To prove this, by the way, one can evaluate T^2, and show that it can be rewritten as a conventional complex Grassmanian integral with matrix Λ. I note by way of correction, that since Λ is not a skew symmetric matrix, the square root of the determinant is not the Pfaffian of Λ. However, one can carry out a unitary transformation from the ψ to a column of real Grassmans (Majoranas), and in this basis a Pfaffian expression can be found, which if $A = \alpha + i\beta$ where α and β are real, skew symmetric matrices,

$$T = pf \begin{pmatrix} i\beta & 2i - i\alpha \\ -2i - i\alpha & -i\beta \end{pmatrix}. \tag{35}$$

which removes the sloppy ambiguity of minus signs in the square root of (34).

(b) For the case where $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, it follows that

$$\Lambda = \left(\begin{array}{cc|cc} 2 & & & a \\ & 2 & -a & \\ \hline & \bar{a} & -2 & \\ -\bar{a} & & & -2 \end{array}\right), \tag{36}$$

so that $\det\Lambda = (4+|a|^2)^2$ and $T = (4+|a|^2)$. You may also verify that if $a = \alpha+i\beta$

$$pf \begin{pmatrix} & i\beta & 2i & -i\alpha \\ -i\beta & & i\alpha & 2i \\ -2i & -i\alpha & & -i\beta \\ i\alpha & -2i & i\beta & \end{pmatrix} = 4 + |a|^2.$$

Let us check this directly. The states of the Hilbert space in an occupation number basis are $|n_1, n_2\rangle$ $(n_i = 0, 1)$. Now $\mathcal{M} = e^{ac_1^\dagger c_2^\dagger} = (1 + ac_1^\dagger c_2^\dagger)$, where, because we are dealing with fermions, the exponential truncates to linear order. This means that

$$\mathrm{Tr}[\mathcal{M}\mathcal{M}^\dagger] = \mathrm{Tr}[(1 + ac_1^\dagger c_2^\dagger)(1 + \bar{a}c_2c_1)] = \mathrm{Tr}[1] + |a|^2 \mathrm{Tr}[c_1^\dagger c_2^\dagger c_2 c_1] = 4 + |a|^2, \tag{37}$$

where the cross terms vanish, as they do not conserve particle number, and we have set $\mathrm{Tr}[1] = 4$, the dimension of the Fock space. This directly confirms that our equation works for the case $N = 2$.