Exercise 1. Path Integrals. (Due Weds 23 Feb.)

1. In this problem consider $\hbar = 1$. Suppose $|0\rangle$ is the ground-state of a harmonic oscillator problem, where $b|0\rangle = 0$. Consider the state formed by simultaneously translating this state in momentum and position space as follows:

$$|p, x\rangle = \exp \left[ -i(x\hat{p} - p\hat{x}) \right] |0\rangle.$$ 

By rewriting $\hat{b} = (\hat{x} + i\hat{p})/\sqrt{2}$, $z = (x + ip)/\sqrt{2}$, show that this state can be rewritten as

$$|p, x\rangle = e^{b^\dagger z - zb}|0\rangle.$$ 

Using the relation $e^{A+B} = e^A e^{B_1[A,B]}$, provided $[A, [A, B]] = [B, [A, B]] = 0$, show that $|p, x\rangle$ is equal to a normalized coherent state

$$|p, x\rangle \equiv |z\rangle e^{-\frac{zz}{2}} = e^{b^\dagger z}|0\rangle e^{-\frac{1}{2}zz}$$

showing that the coherent state $|z\rangle$ represents a minimum uncertainty wavepacket centered at $(q,p)$ in phase space.

2. (a) Suppose $H = \epsilon c^\dagger c$ the Hamiltonian for an energy level $\epsilon$ that can be occupied by a single fermion. Consider the approximation to the partition function obtained by first dividing up the period $\tau \in [0, \beta]$ into $N$ equal time-slices,

$$Z_N = \text{Tr}[(e^{-\Delta \tau H})^N],$$

which is given by

$$Z_N = \int \prod_{j=1}^N d\bar{c}_j dc_j \exp [-S_N] \quad \text{where} \quad S_N = \sum_{j=1}^N \Delta \tau \left[ \bar{c}_j (c_j - c_{j-1}) / \Delta \tau + \epsilon \bar{c}_j c_{j-1} \right].$$

(a) Show that $Z_3$ can be written as a “toy functional integral”,

$$Z_3 = \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \exp \left\{ -(\bar{c}_3, \bar{c}_2, \bar{c}_1) \begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 \\ c_2 \\ c_1 \end{pmatrix} \right\},$$
where $\alpha = 1 - \Delta \tau \epsilon$. In this formula, the discrete time-line is labelled as follows,

$$
\begin{array}{cccc}
\bar{c}_1 & c_2 & c_1 & c_0 = -c_3 \\
\beta = \tau_3 & \tau_2 & \tau_1 & 0
\end{array}
$$

(4)

where $(\bar{c}_j, c_j)$ are the conjugate Grassman variables at each discrete time $\tau_j = j\Delta \tau$.

(b) Evaluate $Z_3$.

(c) Generalize the result to $N$ time slices and obtain an expression for $Z_N$. What is the limiting value of your result as $N \to \infty$?

(d) Repeat the calculation for a boson $H = \epsilon b^\dagger b$

3. Using path integrals, calculate the partition function for a single Zeeman-split electronic level described by the action

$$
S = \sum_{\sigma = \pm 1} \int d\tau \bar{f}_\sigma \left( \partial_\tau - \mu B \sigma B \right) f_\sigma
$$

Why is your answer not the same as the partition function of a spin $S = 1/2$ in a magnetic field?

4. A system of weakly interacting superfluid bosons is described by the action

$$
S = \int_0^\beta d\tau d^3 x \left[ \frac{\hbar^2}{2m} |\nabla \psi|^2 - \mu |\psi|^2 + \frac{g}{2} |\psi|^4 \right].
$$

(5)

(a) Ignoring fluctuations of the Bose Field, sketch the Free energy ($F = S/\beta$) as a function of the magnitude of a uniform Bose field, for $\mu > 0 \mu = 0$ and $\mu < 0$.

(b) Assuming you can ignore fluctuations of the Bose field, when $\mu > 0$, what is the uniform equilibrium value of the Bose field $\psi(x)$?

(c) What is the corresponding coherent state?

(d) Assume that the superfluid lives on a torus. Suppose the phase of the wave-function winds through $l$ turns as one goes around the torus in the $x$ direction. Write down the wavefunction for this state. What is the superfluid velocity of this state?

(e) Why is the superflow persistent?
(f) The above solution is known as a “saddle point” solution. How could one improve
the estimate of the Free energy to take into account fluctuations?

5. (a) (Optional and hard). Suppose

\[ \mathcal{M} = e^{\frac{i}{2} \sum_{i,j} A_{ij} c_i^\dagger c_j^\dagger} \]

where \( A_{ij} \equiv A_{2N} \) is an \( 2N \times 2N \) antisymmetric matrix, and the \( c_j^\dagger \) are a set of
\( 2N \) canonical Fermi creation operators. Using coherent states, calculate

\[ \text{Tr}[\mathcal{M}\mathcal{M}^\dagger] \]

where the trace is over the \( 2^{(2N)} \) dimensional Hilbert space of fermions. (Hint: notice that \( \mathcal{M}\mathcal{M}^\dagger \) is already normal ordered, so that by using the trace formula, you can rewrite this in terms of a Gaussian Grassman integral. You may use the result

\[ \int \prod_{j=1}^{2N} dc_j dc_j^\dagger \exp \left[ -\frac{1}{2} \bar{\Psi} A \Psi \right] = \sqrt{\text{det} \Lambda} = \text{Pf}[A] \]  

where \( \Psi = (c_1, \ldots, c_{2N}, \bar{c}_1, \ldots, \bar{c}_{2N})^T \) is a column vector of Grassmann’s and \( \bar{\Psi} = (\bar{c}_1, \ldots, \bar{c}_{2N}, c_1, \ldots, c_{2N}) \) its Hermitian transpose. Pf denotes the Pfaffian. )

(b) Explicitly check your result for the N=1 case, case \( A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \).