INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2011

Questions 5: Due Weds Nov 12th

1. Consider a gas of particles with interaction

$$\hat{V} = 1/2 \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}\sigma\sigma'} V_{\mathbf{q}} c^{\dagger}{}_{\mathbf{k}-\mathbf{q}\sigma} c^{\dagger}{}_{\mathbf{k}'+\mathbf{q}\sigma'} c_{\mathbf{k}'\sigma'} c_{\mathbf{k}\sigma}$$

   (a) Let $|\phi_0\rangle = \prod_{|\mathbf{k}|<k_F,\sigma} c^{\dagger}{}_{\mathbf{k}\sigma}|0\rangle$ represent the filled Fermi sea, i.e. the ground state of the non interacting problem. Use Wick's theorem to evaluate an expression for the expectation value of the interaction energy $\langle\phi_0|\hat{V}|\phi_0\rangle$ in the non-interacting ground state. Give a physical interpretation of the two terms that arise and draw the corresponding Feynman diagrams. You may use the result

$$\langle\phi|c^{\dagger}{}_{\mathbf{k}\sigma} c_{\mathbf{k}'\sigma'}|\phi\rangle = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}\theta(k_F - k) \qquad (1)$$

   (b) Draw the all the Feynman diagrams corresponding to the second order corrections to the ground-state energy.

   (c) Discuss how to convert one or more of the above Feynman diagrams into a mathematical expression, using the Feynman rules.

2. The separation of electrons $R_e$in a Fermi gas is defined by

$$\frac{4\pi R_e^3}{3} = \rho^{-1}$$

   where ρ is the density of electrons. The dimensionless separation r_s is defined as $r_s = R_e/a$ where $a = \frac{c\hbar^2}{me^2}$ is the Bohr radius.

   (a) Show that the Fermi wavevector is given by

$$k_F = \frac{1}{\alpha r_s a}$$

   where $\alpha = \left(\frac{4}{9\pi}\right)^{\frac{1}{3}} \approx 0.521$.

   (b) Consider an electron plasma where the background charge density precisely cancels the charge density of the plasma. Show that the ground-state energy to leading order in the strength of the Coulomb interaction is given by

$$\begin{aligned} \frac{E}{\rho V} &= \frac{3}{5}\frac{R_Y}{\alpha^2 r_s^2} - \frac{3}{4\pi}\frac{R_Y}{\alpha r_s} \\ &= \left(\frac{2.21}{r_s^2} - \frac{0.912}{r_S}\right) R_Y \end{aligned} \qquad (2)$$

   where $R_Y = \frac{\hbar^2}{2ma^2}$ is the Rydberg energy. (Hint - in the electron gas with a constant charge background, the Hartree part of the energy vanishes. The Fock part is the second term in this expression. You may find it useful to use the integral

$$\int_0^1 dx \int_0^1 dy xy \ln|\frac{x+y}{x-y}| = \frac{1}{2}$$

(c) When can the interaction effects be ignored relative the kinetic energy?

3. Consider a Fermi gas containing Fermions with spin degeneracy $n_s = 2S+1 > 2$,and total density $n_s\rho$, where ρ is the density per spin component. Nucleons can be considered as the special case where $n_s = 4$, corresponding to spin and isospin quantum number.

   (a) Suppose the fermions interact via a repulsive three body interaction:

   $$V(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) = \beta\delta^{(3)}(\mathbf{r}_i - \mathbf{r}_j)\delta^{(3)}(\mathbf{r}_j - \mathbf{r}_k), \tag{3}$$

   Write this interaction in second-quantized form.

   (b) Invent a symbol for a three body interaction and write down the Feynman rules.

   (c) Use your rules to calculate the interaction energy per unit volume, $V(n_s, \rho)$ to leading order in α. What happens when $n_s = 1$ or $n_s = 2$?

   (d) If we neglect Coulomb interactions, why is the case $n_s = 4$ relevant to nuclear matter?