

INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2010

Questions 4. (Due Weds, Oct 26th)

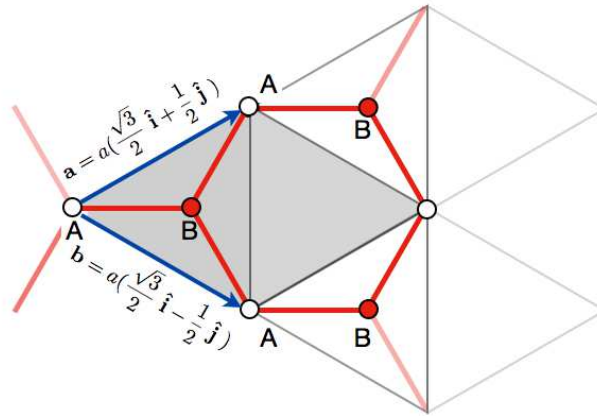


Figure 1: Honeycomb structure of graphene. See Problem 1

1. Electrons on graphene move on a Honeycomb lattice as shown in Fig. (1). The vertices of each unit cell form a triangular lattice of side length  $a$ , located at positions  $\mathbf{r}_i = m\mathbf{a} + n\mathbf{b}$ , where  $\mathbf{a} = a\left(\frac{\sqrt{3}}{2}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}\right)$  and  $\mathbf{b} = a\left(\frac{\sqrt{3}}{2}\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}\right)$  are the lattice vectors. There are two atoms per unit cell, labelled “A” and “B”. In a simplified model of graphene, electrons can occupy  $\pi$  orbitals at either the A or the B sites, with a tight-binding hopping matrix element  $-t$  between neighboring sites.

- (a) Construct a tight-binding model for graphene. For simplicity, ignore the spin of the electron. Suppose the creation operator for an electron in the A or B orbital in the “ $i$ ”th cell is  $\psi^\dagger_A(\mathbf{r}_i)$ . Show that the tight-binding Hamiltonian can be written in the form

$$H = -t \sum_j \left\{ \left[ \psi^\dagger_B(\mathbf{r}_i) + \psi^\dagger_B(\mathbf{r}_i - \mathbf{a}) + \psi^\dagger_B(\mathbf{r}_i - \mathbf{b}) \right] \psi_A(\mathbf{r}_i) + \text{H.c.} \right\} + (\epsilon - \mu) \sum_i (n_A(i) + n_B(i))$$

where  $\epsilon$  is the energy of a localized orbital.

- (b) By transforming to momentum space, writing

$$\psi^\dagger_\lambda(\mathbf{r}_j) = \frac{1}{\sqrt{N_s}} \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\lambda} e^{-i\mathbf{k}\cdot\mathbf{r}_j} \quad (\lambda = A, B)$$

and  $N_s$  is the number of unit cells in the crystal, show that the Hamiltonian can be written

$$H = \sum_{\mathbf{k}} \left( c^\dagger_{\mathbf{k}B}, c^\dagger_{\mathbf{k}A} \right) \begin{bmatrix} \epsilon - \mu & \Delta(\mathbf{k}) \\ \Delta^*(\mathbf{k}) & \epsilon - \mu \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}B} \\ c_{\mathbf{k}A} \end{pmatrix}$$

where

$$\Delta(\mathbf{k}) = -t(1 + e^{i\mathbf{k}\cdot\mathbf{a}} + e^{i\mathbf{k}\cdot\mathbf{b}})$$

with energy eigenstates

$$\epsilon(\mathbf{k}) = \pm|\Delta(\mathbf{k})| + (\epsilon - \mu).$$

- (c) Show that  $\Delta(\mathbf{k}) = 0$  at two points in the Brillouin zone where  $\mathbf{k}\cdot\mathbf{a} = -\mathbf{k}\cdot\mathbf{b} = \pm\frac{2\pi}{3}$ , given by

$$\mathbf{k} = \pm\mathbf{K}$$

where  $\mathbf{K} = \frac{4\pi}{3a}\hat{\mathbf{j}}$ .

- (d) By expanding around  $\mathbf{k} = \pm\mathbf{K} + \mathbf{p}$ , showing that when  $\mathbf{p}$  is small,  $\Delta_{\mathbf{p}\pm\mathbf{K}} = \pm\tilde{c}(p_y \pm ip_x)$ , where  $\tilde{c} = \frac{\sqrt{3}}{2}at$  is a “renormalized” speed of light. By defining a spinor for the two cones

$$\psi_{\mathbf{p}+} = \begin{pmatrix} c_{\mathbf{p}+\mathbf{K}\mathbf{B}} \\ c_{\mathbf{p}+\mathbf{K}\mathbf{A}} \end{pmatrix}, \quad \psi_{\mathbf{p}-} = \begin{pmatrix} c_{\mathbf{p}-\mathbf{K}\mathbf{A}} \\ -c_{\mathbf{p}-\mathbf{K}\mathbf{B}} \end{pmatrix},$$

show that the low energy Hamiltonian can be written as a Dirac equation

$$H = \sum_{\mathbf{p}\lambda=\pm} \psi_{\mathbf{p}\lambda}^\dagger ((\vec{\sigma} \times \mathbf{p}) + (\epsilon - \mu)\mathbf{1}) \psi_{\mathbf{p}\lambda}$$

where  $\vec{\sigma}$  is a Pauli pseudo-spin matrix acting in the two-component sublattice space, so that when  $\epsilon - \mu = 0$ , the excitation spectrum is defined by two Dirac cones with  $E(\mathbf{p}) = \pm\tilde{c}p$ .

2. Suppose we turn on the interaction adiabatically, writing  $V_{int}(\lambda) = \lambda\hat{V}$ . It is possible to relate the change in the Free energy  $\Delta F$  to the interaction energy averaged over interaction strength with weight  $1/\lambda$ . Prove this using the following outline.

- (a) Consider the Free energy of the system, which is given by  $F(\lambda) = -T \ln \text{Tr}[e^{-\beta[H_0 + \lambda V]}]$ . First show by differentiating this result that

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{\lambda} \langle V_{int}(\lambda) \rangle \quad (1)$$

- (b) Use this result to reason that the change in the free energy as a result of turning on the interaction is given by

$$\Delta F = \int_0^1 \frac{d\lambda}{\lambda} \langle V_{int}(\lambda) \rangle \quad (2)$$

By taking the zero temperature limit of this expression, we can reason that

$$\Delta E = \int_0^1 \frac{d\lambda}{\lambda} \langle \phi(\lambda) | V_{int}(\lambda) | \phi(\lambda) \rangle \quad (3)$$

where  $|\phi(\lambda)\rangle$  is the ground-state for  $H(\lambda) = H_0 + \lambda\hat{V}$ . In otherwords, the change in energy is equal to the interaction energy, averaged over the interaction strength, with weight  $1/\lambda$ .

- (c) If the interaction energy has an expansion  $\langle V_{int}(\lambda) \rangle = \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 + \dots$ , what is the corresponding expression for the change in the ground-state energy in increasing  $\lambda$  to full strength  $\lambda = 1$ ?

- (d) Why is the change in the ground-state energy not equal to the change in the interaction energy? (Hint- as the interaction increases- what is changing, other than the Hamiltonian? )

3. Construct the diagram technique for the one-particle scattering amplitude on the potential  $U(x)$ .

$$\begin{aligned}
 \text{1} \bullet \text{2} &= \text{1} \times \text{2} + \text{1} \times \xrightarrow{k''} \times \text{2} + \text{1} \times \xrightarrow{k''} \times \xrightarrow{k'''} \times \text{2} + \dots \\
 &= \text{1} \times \text{2} + \text{1} \times \xrightarrow{k''} \bullet \text{2}
 \end{aligned}$$

- (a) What is denoted by the crosses and the solid line?
- (b) Show that the electron Green's function has the form

$$G_{\vec{k},\vec{k}'}(E) = G_{\vec{k}}^{(0)}(E)\delta_{\vec{k},\vec{k}'} + G_{\vec{k}}^{(0)}(E)t_{\vec{k},\vec{k}'}(E)G_{\vec{k}}^{(0)}(E) \tag{4}$$

where the scattering amplitude satisfies the integral equation

$$t_{\vec{k},\vec{k}'}(E) = U(\vec{k} - \vec{k}') + \int \frac{d^d q}{(2\pi)^d} \frac{U(\vec{k} - \vec{q})}{E - E(q) + i\delta} t_{\vec{q},\vec{k}'}(E) \tag{5}$$

- (c) Evaluate the t-matrix for a delta-function potential  $U(x) = U\delta^{(d)}(x)$ . (Hint: the t-matrix is momentum independent).
- (d) Show that in dimensions  $d \leq 2$ , a bound-state forms beneath the bottom of the conduction band for arbitrarily weak attractive scattering potential. (Hint: this means checking to see if there are negative energy poles in the t-matrix )