

INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2011

Questions 3. (Due Weds , Oct 12th)

1. (a) Show that for a general system of conserved particles at chemical potential, the total particle number in thermal equilibrium can be written as $N = -\partial F/\partial\mu$, where $F = -k_B T \ln Z$ and

$$Z = \text{Tr}[e^{-\beta(\hat{H}-\mu N)}]. \quad (1)$$

- (b) Apply this to a single bosonic energy level, where

$$H - \mu N = (\epsilon - \mu)a^\dagger a \quad (2)$$

and \hat{a}^\dagger creates either a Fermion, or a boson, to show that

$$\begin{aligned} F &= \pm k_B T \ln[1 \mp e^{-\beta(\epsilon-\mu)}] \\ \langle \hat{n} \rangle &= \frac{1}{e^{\beta(\epsilon-\mu)} \mp 1} \end{aligned} \quad (3)$$

where the upper sign refers to bosons, the lower, to fermions. Sketch the occupancy as a function of ϵ for the case of fermions and bosons. Why does μ have to be negative for bosons?

2. Bose Einstein condensates created inside optical atom traps contain alkali atoms at densities of about $10^{14} - 10^{15} \text{cm}^{-3}$.

- (a) What is the Bose Einstein transition temperature of a gas of Sodium atoms at a density 10^{15}cm^{-3} ? (Give your answer in micro-Kelvin.) How are such temperatures attained in practice? (See <http://cua.mit.edu/ketterle-group> for more information.)
- (b) Liquid Helium has a density of 122g/litre at its boiling point. Compare its theoretical Bose Einstein condensation temperature with its superfluid transition temperature (2.21 K). Why are the two numbers not the same?

3. Consider a system of fermions or bosons, created by the field $\psi^\dagger(\vec{r})$ interacting under the potential

$$V(r) = \begin{cases} U, & (r < R), \\ 0, & (r > R), \end{cases} \quad (4)$$

- (a) Write the interaction in second quantized form.

- (b) Switch to the momentum basis, where $\psi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} c_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$. Verify that $[c_{\vec{k}}, c_{\vec{k}'}^\dagger]_{\pm} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$, and write the interaction in this new basis. Please sketch the form of the interaction in momentum space.

4. (Equivalence of the micro and grand canonical ensembles in large systems.)

In a microcanonical ensemble, the density matrix can be given by

$$\hat{\rho}_M = \frac{1}{W} \delta(E - \hat{H}) \delta(N - \hat{N})$$

where E and N are the energy and particle number respectively, while

$$W \equiv W(E, N) = \text{Tr} \left[\delta(E - \hat{H}) \delta(N - \hat{N}) \right]$$

is the “density of states” at energy E , particle number N . This normalizing quantity plays a role similar to the partition function in the Gibb’s ensemble.

(a) By rewriting the delta functions inside the above trace W as an inverse Laplace transform,

$$\begin{aligned} \delta(x - \hat{H}) &= \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{d\beta}{2\pi i} e^{-\beta(x - \hat{H})}, \\ \delta(x - \hat{N}) &= \int_{\zeta_0 - i\infty}^{\zeta_0 + i\infty} \frac{d\zeta}{2\pi i} e^{\zeta(x - \hat{N})}, \end{aligned} \quad (5)$$

and evaluating the resulting integrals at the saddle point of the integrand, show that for a large system W is related to the entropy by Boltzmann’s relation

$$S(E, N) = k_B \ln W(E, N).$$

(b) Using your results, show that in a large system, the expectation value of an operator is the same for corresponding Grand and microcanonical ensembles, namely

$$\langle A \rangle = \text{Tr}[\rho_M \hat{A}] = \text{Tr}[\rho_B \hat{A}]$$

where $\hat{\rho}_B = Z^{-1} e^{-\beta(H - \mu \hat{N})}|_{\beta=\beta_0, \mu=\mu_0}$ is the Boltzmann density matrix evaluated at the saddle point values of β_0 and μ_0 ,

$$\beta_0 = \frac{\partial \ln W}{\partial E}, \quad \mu_0 = \beta_0^{-1} \frac{\partial \ln W}{\partial N}.$$