Last time we started our detailed discussion of SU(2) by noting that by selecting $L_3$ to be the generator of the one-dimensional Cartan subalgebra and observing that $L_\pm = \dfrac{L_1 \pm iL_2}{\sqrt{2}}$ are raising and lowering operators, meaning that if we have any eigenstate $e_m$ of $L_3$ with eigenvalue $m$, we have eigenstates $L_\pm e_m$ with eigenvalues $m \pm 1$ unless the operator annihilates the state $e_m$. As each state so generated gives a new dimension to the vector space, if the representation is to be finite dimensional there must be a highest weight state with $m = m_{\rm max} =: j$ and another with lowest $m$ value, which we found to be $m_{\rm min} = -j$. Writing the states so generated as $|j, m\rangle$, we found the representations, one for each $j$ with $2j \in \mathbb{N}$, with

$$\begin{aligned} L_3\,|j,m\rangle &= m\,|j,m\rangle \\ L_+\,|j,m\rangle &= \frac{1}{\sqrt{2}}\sqrt{(j-m)(j+m+1)}\,|j,m{+}1\rangle \\ L_-\,|j,m\rangle &= \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\,|j,m{-}1\rangle \end{aligned}$$

We then began our discussion of decomposing the tensor product of two irreducible representations into a direct sum of irreducible representations. This is important in physics because when a total state can be considered as having two noninteracting or nearly noninteracting components, the full state transforms under symmetries as the tensor product of the pieces. Choosing to diagonalize $L_3$ we saw that each finite dimensional representation has a "highest weight" state with the eigenvalue $m$ of $L_3$ we call $j$. We saw that there is a unique (up to equivalence) irreducible representation for each $j$ equal to a nonnegative integer or half-integer. Then we began looking at the tensor product $\Gamma^{j_1} \otimes \Gamma^{j_2}$ and observed that the highest $m$ is $j_1 + j_2$, so that is the highest $j$ in the representation.

## Today

We will continue to examine the $m$ values in the tensor product, to conclude that

$$\Gamma^{j_1} \otimes \Gamma^{j_2} \cong \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \Gamma^j,$$

that is, we get a piece for each $j$ between the difference of the two $j$'s and the sum, though of course only integer $j$'s if $j_1 + j_2$ is an integer, and only integer plus $\frac{1}{2}$ otherwise.

When the pieces of a quantum mechanical wave function do interact in a way obeying the symmetry, each irreducible component in the direct sum will be affected separately, so it is important to be able to project the pieces of the direct product onto the individual states in the direct sum. This overlap is what physicists call Clebsch-Gordon coefficients, or less ambiguously, Vector coupling constants.

We will give the procedure for finding those coefficients, starting a simple example of $(j = 1) \otimes (j = \frac{1}{2})$, which you will finish up for homework. Of course when you really need Clebsch-Gordon coefficients you can find them from the Particle Data Group (`pdg.lbl.gov`) or Wikipedia, but you should have worked one set out for yourself.

We will consider the representations of the finite group elements and how these relate to the spherical harmonics $Y_\ell^m(\theta, \phi)$ and the Wigner-Eckhart theorem. This determines much of how transitions in nuclei or atomic physics occur when a photon is released, for example.

We will then turn to another SU(2) group, that of isospin in nuclear and particle physics. There is an approximate symmetry under which protons and neutrons can be viewed as interchangable, and hadronic states then form irreducible representations of the SU(2) which involves rotations in the abstract two-dimensional space formed by this pair of states. Nowadays this is just a small part of flavor symmetry, but it is a much better symmetry than the rest, and it is significant in nuclear physics that nuclei form multiplets under this group, and in low-energy high-energy physics as well, as we shall see.

## Reminders:

Homework 4 is due Thursday, Feb. 16.

I will try to get Homework 5 posted before Thursday.