Last week we found that the connected compact Lie group elements are described by exponentials of the generators $\{L_i\}$, and at least the local structure by the **structure constants** ${c_{ij}}^k$, with

$$[L_i, L_j] = i \sum_k {c_{ij}}^k L_k,$$

This closure under commutation, and the fact that commutators obey the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

mean the generators are the basis of a **Lie algebra**. Any representation of the group is also a representation of the algebra.

On Tuesday, after looking at some simple (in the ordinary sense as well as the mathematical sense) groups and finding their structure constants, we discussed the global topological property of simply connectedness, finding that SO(3) was not but SU(2) was, describing their manifolds (but not their metrics).

Then we defined the adjoint representation of any Lie algebra, given by the structure functions, and defined the Killing form, a bilinear real-valued map on $\mathcal{L} \times \mathcal{L}$, which, by suitable choice of the basis vectors for the algebra, can be made diagonal with eigenvalues $\pm 1$ or 0. We defined simplicity and semisimplity, and we have theorems that say the group is semisimple if and only if all eigenvalues are not zero, and is compact only if all are positive.

Finally we discussed the Lorentz and Poincaré groups, the first being semisimple but not compact, the second not semisimple. The noncompactness also means the finite dimensional representations may not be unitary, and indeed this is why fermion fields $\psi$ are coupled to $\bar{\psi}$ rather than $\psi^\dagger$ in the Lagrangian,

## Today

We will briefly discuss how symmetries are interpreted on quantum operators, which will polish off this chapter. But the real topic for today is working out thoroughly the most important Lie group for physics, SU(2).

We will begin by finding all the finite-dimensional representations. Hopefully you already know from quantum mechanics that there is a single representation for each dimension, labelled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and for the $j'^{\text{th}}$ representations there is a single state for each $L_z = m$ for each $m \in [-j, j]$ with $j - m \in \mathbb{Z}$. We will prove this, with a technique we will later use for more complicated groups. In the process we will introduce the concepts of the Cartan subalgebra, and the weights of states in the representation.



We will find the explicit representations of SU(2) and then address direct products of irreducible representations and their reduction to sums of irreducible representations, which will tell us about what physicists call Clebsch-Gordon coefficients. Because combining the angular momenta of particles in a compound object like a nucleus or an atom is so important, there is an impressive array of machinery developed for this, to which I will give some references.

Because the SU(2) group is used to discuss physical rotations in space, we pay attention to the representation of finite group elements as well as the generators, which we will not do in general for bigger groups. We will probably not get to this today.

## Reminders:

Homework 4 has been posted and is due next Thursday, Feb. 16.