On Tuesday we found that connected compact Lie group elements are described by exponentials of the generators \( \{ L_i \} \), and at least the local structure by the structure constants \( c_{ij}^k \), with

\[
[L_i, L_j] = i \sum_k c_{ij}^k L_k,
\]

This closure under commutation, and the fact that commutators obey the Jacobi identity

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0
\]

mean the generators are the basis of a Lie algebra. Any representation of the group is also a representation of the algebra. In particular, we saw that

\[
\Gamma_{jk}^{adj}(L_i) = i c_{ji}^k,
\]

is the adjoint representation. We then defined the Killing form

\[
\beta_{ij} = \beta(L_i, L_j) = \text{Tr} \left( \Gamma_{jk}^{adj}(L_i) \Gamma_{kj}^{adj}(L_j) \right) = -\sum_{ab} c_{ai}^b c_{bj}^a.
\]

and I claimed that \( \beta \) is nonsingular if and only if the Lie algebra is semisimple (has no invariant abelian subalgebras), and I will claim it is positive definite if the group is also compact, in which case the finite dimensional representations are unitary.

We looked at two examples of groups, SO(3) and SU(2), with the same Lie algebra and \( c_{ij}^k = \epsilon_{ijk} \), so \( \beta_{ij} = 2 \delta_{ij} \) and both groups are semisimple (in fact simple) and compact. We then looked at the Poincaré group, and found it to be not semisimple, as the momenta formed an abelian ideal, and the Lorentz subgroup is not compact. This is the reason the finite dimensional representations are not unitary, and why we need \( \bar{\psi} \) instead of \( \psi^\dagger \) for fermion fields in quantum field theory.

Today

We will briefly discuss how symmetries are interpreted on quantum operators, which will polish off this chapter. But the real topic for today is working out thoroughly the most important Lie group for physics, SU(2).

We will begin by finding all the finite-dimensional representations. Hopefully you already know from quantum mechanics that there is a single representation for each dimension, labelled by \( j \), and for the \( j^{th} \) representations there is a single state for each \( L_z = m \) for each \( m \in [−j, j] \) with \( j−m \in \mathbb{Z} \). We will prove this, with a technique we will later use for more complicated groups. In the process we will introduce the concepts of the Cartan subalgebra, and the weights of states in the representation.

We will find the explicit representations of SU(2) and then address direct products of irreducible representations and their reduction to sums of irreducible representations, which will tell us about what physicists call Clebsch-Gordon coefficients. Because combining the angular momenta of particles in a compound object like a nucleus or an atom is so important, there is an impressive array of machinery developed for this, to which I will give some references.

Because the SU(2) group is used to discuss physical rotations in space, we pay attention to the representation of finite group elements as well as the generators, which we will not do in general for bigger groups. We will probably not get to this today.

Reminders:

I misspoke last time. Next week we meet only on Tuesday, the following week we meet on Tuesday, Wednesday and Friday.

Homework 4 has been posted and is due next Thursday, Feb. 18.