Review

Yesterday we proved the Great Orthogonality Theorem

$$\sum_{A \in G} \Gamma_{ab}^i(A) \Gamma_{cd}^j(A) = \frac{g}{\ell_i} \delta_{ij} \delta_{ac} \delta_{bd},$$

which tells us that the space of complex linear functions on the group has an orthonormal basis given by the complete set of inequivalent irreducible representations of the group. As a consequence,

$$g = \sum_i \ell_i^2.$$ 

We then defined characters of a representation as the trace of the representative,

$$\chi^i(A) = \text{Tr} \, \Gamma^i(A),$$

which are functions $G \to \mathbb{C}$, but as $\chi^i$ takes the same value on conjugate elements of the group, is really a function from the conjugacy classes, so we have $\tilde{\chi}^i : \{C_k\} \to \mathbb{C}$. In fact they are orthonormal functions (with a suitable weight $\eta_k$ on the classes) from $\{C_k\} \to \mathbb{C}$, which tells us there can’t be more irreducible representations than there are conjugacy classes. The orthonormality gives us a method of finding out how many times $a_i$ each irreducible representation $\Gamma^i$ occurs in a given representation $\Gamma$:

$$a_j = \frac{1}{g} \sum_{A \in G} \chi^i(A) \chi^j(A).$$

In particular, if $\Gamma$ is irreducible only one $a_j \neq 0$ and it is 1, so $\Gamma$ is irreducible if and only if $\sum_{A \in G} \chi^i(A) \chi^j(A) = g$.

We then defined the regular representation

$$\Gamma_{\text{reg}}^{\alpha \beta}(B) = \delta_{A_{\alpha} B, A_{\beta}},$$

We found that each irreducible representation $\Gamma^i$ occurs $\ell_i$ times in its reduction, a fact we already knew from earlier.

Today

We will examine arbitrary linear functions $f : \{C_k\} \to \mathbb{C}$ which are, of course, also functions in $L_G : G \to \mathbb{C}$ and therefore expandable in the representation matrix elements. The coefficients in that expansion will give us matrices which commute with each irreducible representation, which therefore (by Schur’s first) are multiples of the identity, and provide the coefficients for expanding $f$ in the characters, which shows the number of conjugacy classes is not greater than the number of irreducible elements. So together with the above,

the number of representations is equal to the number of conjugacy classes.

We will use this information to find all the representations of $C_{4v}$. Then we will see how this might help in finding the eigenfunctions of a Hamiltonian with $C_{4v}$ symmetry.

I will barely mention crystallographic groups, point groups and space groups which occur in crystallography.

The chapter ends with tensor products of representations, their decomposition into irreducible representations and Clebsch-Gordan coefficients.

Next, we turn to the main section of the course, the discussion of infinite groups, in particular we focus on topological groups.

Reminders:

We have class Friday as usual, in 203 at noon.

Consider going to the colloquium today at 4:45 by Peter Woit, on how mathematics might tell us about physics in ways experiment cannot. He has been involved in overseeing undergraduate math courses and is currently teaching a course that sounds much like this one, though somewhat more advanced. Cookies at 4:30.

Homework 2 is due on tomorrow, Feb. 4, by 4:00 PM. Homework 3, due Feb. 11, has been posted.