Physics 618 Fifth Lecture Jan. 31, 2017

Review

Last time we proved Schur's Second Lemma and the **Great Orthogonality Theorem**

$$\sum_{A\in G} \Gamma^i_{ab}(A)\Gamma^{j*}_{cd}(A) = \frac{g}{\ell_i}\delta_{ij}\delta_{ac}\delta_{bd},$$

which tells us that the space of complex linear functions on the group has an orthonormal basis given by the complete set of inequivalent irreducible representations of the group. As a consequence,

$$g = \sum_i \ell_i^2.$$

In the course of proving this, we defined how the elements of the group act on the space of functions on the group (into $\mathbb{C}$):

$$(B\psi)(A) = \psi(AB).$$

This led to the regular representation

$$\Gamma^{\text{reg}}_{\alpha\beta}(B) = \delta_{A_\alpha B,\, A_\beta}.$$

which is a g-dimensional reducible representation.

We then defined **characters** of a representation as the trace of the representative,

$$\chi^i(A) = \operatorname{Tr}\Gamma^i(A),$$

which are functions $G \to \mathbb{C}$, but as χ^i takes the same value on conjugate elements of the group, is really a function from the **conjugacy classes**, so we have $\tilde{\chi}^i : \{C_k\} \to \mathbb{C}$. In fact they are orthonormal functions (with a suitable weight η_k on the classes) from $\{C_k\} \to \mathbb{C}$, which tells us there can't be more irreducible representations than there are conjugacy classes. The orthonormality gives us a method of finding out how many times a_i each irreducible representation Γ^i occurs in a given representation Γ:

$$a_j = \frac{1}{g}\sum_{A\in G} \chi^{j*}(A)\chi(A).$$

In particular, if Γ is irreducible only one $a_j \neq 0$ and it is 1, so Γ is irreducible if and only if $\sum_{A\in G} \chi^*(A)\chi(A) = g$.

Today

By looking at the characters of the regular representation we will find that each irreducible represention of dimension ℓ_i occurs ℓ_i times in reducing the regular representation, and this tells us that the characters form a orthogonal *basis* on the conjugacy classes (weighted by the number of elements in each class), and thus that

> The number of irreducible representations is equal to the number of conjugacy classes.

We will use this information to find all the representations of C_{4v}.

If we still have time, we will see how this might help in finding the eigenfunctions of a Hamiltonian with C_{4v} symmetry.

Then I will barely mention crystallographic groups, point groups and space groups which occur in crystalography.

The chapter ends with tensor products of representations, their decomposition into irreducible representations and Clebsch-Gordon coefficients, though it is unlikely we will get that far today.

Next time we will turn to the main section of the course, the discussion of infinite groups, in particular we focus on topological groups.

Reminders:

- Homework 2 is due Thursday, Feb. 2, by 4:00 PM.
- Homework 3, due Feb. 9, has been posted.