Physics 618 Twenty-second Lecture April 11, 2017

Last time we defined three kinds of derivative operators on the group manifold, or rather on functions $f(U)$ defined on the group manifold. As each group element can be written as $U = e^{i\omega^i L_i}$, the ordinary partial derivative with respect to the parameters ω^i is

$$\Pi_j = \frac{\partial f(e^{i\omega^k L_k})}{\partial \omega^j}.$$

But we also defined the left-derivative and the right-derivative:

$$E_j f(U) = -i\,\frac{\partial}{\partial \omega^j} f\left(e^{i\omega^j L_j} U\right)\Big|_{\omega=0}; \qquad R_j f(U) = -i\,\frac{\partial}{\partial \omega^i} f\left(U e^{i\omega_j L_j}\right)\Big|_{\omega=0}.$$

We also represented E_j and R_j as matrices $R_j = R_{ab}(L_j)_{ab}$ with $E_{ab} = U_{ac}\frac{\partial}{\partial U_{bc}}$, $R_{ab} = U_{cb}\frac{\partial}{\partial U_{ca}}$. Thus we have $Rf = U^{-1}(Ef)U$. We found that the relation which conjugates the Lie algebra generators by group elements

$$e^{-i\omega^k L_k} L_\ell e^{i\omega^k L_k} = S_{\ell m}(e^{i\omega^b L_b}) L_m.$$

where $S_{\ell m}(U)$ is the adjoint representation of the group, with $S(e^{i\omega^j L_j}) = \exp i\omega^j \mathcal{S}_{..}$, where $\mathcal{S}_{..}$ is the adjoint representation of the Lie algebra. We then saw that S provides the linear transformation from R to E, $E_m = S_{m\ell} R_\ell$.

More important is the connection between E_ℓ, the square of which is what enters the Hamiltonian, and the momenta Π_ℓ which are the (commuting) ordinary partial derivatives. I think this is where I left off last time, at least where I stopped being thorough, so

Today

we will examine that connection, using the relation you proved for homework,

$$\frac{\partial}{\partial x} e^{A(x)} = \int_0^1 d\alpha\, e^{\alpha A(x)}\, \frac{\partial A(x)}{\partial x}\, e^{(1-\alpha)A(x)}.$$

Consider the neighborhood of the group manifold around $U_0 = e^{i\omega_0^\ell|_\ell}$ with elements U given in two ways,

$$U(\omega) = e^{i\omega^\ell L_\ell} = e^{i\rho^\ell L_\ell} e^{i\omega_0^\ell L_\ell}$$

The left-derivative is differentiating with respect to ρ, while Π is with respect to ω, so the relation involves the partial-derivative matrix $\partial\rho^m/\partial\omega^\ell$.

Why does all this matter? We are interested in finding the invariant volume element, for which the measure of a infinitesimal region R' which is the image of a region R under $U' = GU$ for some fixed G, has the same volume as R does. Clearly $d^n\rho$ does this for R an infinitesimal region about the identity, and our normalization of the Killing form as $\beta(L_j, L_k) = \delta_{jk}$ says that $\mu(\nu) \prod d\nu = \prod d\rho$. So

$$\mu\nu = \left| \frac{\partial \rho^m}{\partial \omega^\ell} \right| .$$

This measure is called the Haar[1] or Hurwitz[2] measure. If the group elements are coordinates in a quantum field theory, the path integral interpretation says we are to integrate over all possible values, but we need to do so with a measure that respects the group invariance, so that means using the Hurwitz measure. We will find that the measure for any group element can be found in terms of the eigenvalues of that group element in the adjoint representation. This is particularly simple for $SU(2)$, and we will see that the correct measure is just that of integrating over S_3, a 3-sphere in four-dimensional space.

[1] Alfréd Haar, 1933.

[2] Adolf Hurwitz, 1897, for Lie groups.