Last time we went through considerable algebra to relate the ordinary derivatives of functions on the group, \( \partial f(e^{i\omega L_j})/\partial \omega^k \) to the right handed derivatives \( \partial f(e^{i\omega L_j}e^{i\nu L_j})/\partial \nu^k \bigg|_{\nu^0=0} \). We found that if \( e^{i\omega^\ell L_\ell} = e^{i\rho^\ell L_\ell} e^{i\omega^\ell L_\ell} \) relates the coordinates \( \omega \) and \( \rho \) for fixed \( \omega^0 \),

\[
\frac{\partial \rho^m}{\partial \omega^\ell} \bigg|_{\rho^0=0} = \left[ \frac{1 - e^{-i\omega^k S(L_k)}}{i\omega^k S(L_k)} \right]_{\ell m},
\]

and the ordinary derivative \( \partial / \partial \omega^\ell \) is given by the same matrix multiplying \( E_m \).

Today

The same kind of considerations will enable us to find a measure for integration over the group which is invariant under group multiplication. We will find that the appropriate integration measure is \( h(\omega) \prod d\omega^j \) where

\[
h(\omega) \propto \det \left( \frac{e^{i\omega^j S(L_j)} - 1}{i\omega^j S(L_j)} \right).
\]

In particular, if \( G = SU(2) \), the integration over the group is given by

\[
\int_0^{2\pi} 4 \sin^2 \left( \frac{|\omega|}{2} \right) \, d\omega \, d^2\Omega
\]

with \( \Omega = \sin \theta \, d\theta \, d\phi \) the integration over directions of the rotation axis, and \( |\omega| \) the angle of rotation. It is isomorphic to the 3-sphere \( S^3 \) (the surface of a 4-dimensional ball).

After that, we will have a very brief excursion into solid state physics, where our lattice is a crystal. We will first discuss the motion of the atoms of the lattice, and then the effect on electron wave functions of lying in a periodic potential. Discrete translational invariance pushes us to fourier transformed fields, and we discuss free phonons and the constraint on interactions. We also note that electrons, which are not attached to the lattice points, have wave functions described by Block functions which are periodic, though the wave function itself is not. The non-interacting states are described by momenta confined to a Brillouin zone, though there may be many states for a given \( \vec{k} \).

\(^1\)This is a left derivative, while today the notes want the right derivative. The corresponding formula gives the complex conjugate, but as \( S \) is imaginary, this is the same thing.
Returning to the lattice degrees of freedom, we consider the possibility that they have a global symmetry group in the Hamiltonian, but that the symmetry is spontaneously broken. This will lead us to massless Goldstone bosons or spin waves.

Next Time

We previously found local or gauge symmetries require gauge fields which are massless vectors living in the Lie algebra of the symmetry group. Today we will find massless scalar particles, Goldstone bosons, living in a coset space $\mathfrak{G}/\mathfrak{K}$ of the Lie algebra $\mathfrak{G}$ of a global Hamiltonian symmetry by the Lie algebra $\mathfrak{K}$ of its subgroup which preserves the vacuum state. Next time we will see how a gauged symmetry with scalar fields which break the symmetry can, by the Higgs effect, permit the massless gauge bosons to eat the Goldstone bosons and become massive vector particles. This is what makes the $W^\pm$ and the $Z^0$ of Salam-Weinberg possible.