On Tuesday, having completed what we wished to cover on representations of compact simple Lie groups, we turned to field theory. We saw that physics could be described by a set of fields $\eta_i(\vec{r},t)$ and be determined by a Lagrangian density $\mathcal{L}(\eta_i, \vec{\nabla}\eta_i, \dot{\eta}_i; \vec{r}, t)$ where the dynamical variables are only the η_i, while $\vec{r}$ is essentially an index on the dynamical variables. Because the field equations come from varying the action, which is an integral over space-time, it is convenient to introduce the fourth component $x^0 = ct$ even when we don't have a relativistic theory. With the partial derivative operator $\partial_\mu := \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{\partial}{c\partial t}, \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)$ and the notation $B_{,\mu} := \partial B/\partial x^\mu$, the variation of the Lagrangian density is

$$\delta\mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial\eta_i}\delta\eta_i + \sum_i\sum_{\mu=0}^{3}\frac{\partial\mathcal{L}}{\partial\eta_{i,\mu}}\delta\eta_{i,\mu},$$

and when we integrate this over $\int d^4x$, integrating the second term by parts and discarding surface terms, we find

$$\delta I = \sum_i \int \left[\frac{\partial\mathcal{L}}{\partial\eta_i} - \sum_{\mu=0}^{3}\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\eta_{i,\mu}}\right)\right]\delta\eta_i d^4x,$$

As the equations of motion come from insisting the variation of I vanishes for any $\delta\eta_i$, they are

$$\sum_\mu \frac{d}{dx^\mu}\frac{\partial\mathcal{L}}{\partial\eta_{i,\mu}} - \frac{\partial\mathcal{L}}{\partial\eta_i} = 0.$$

The simplest Lagrangian we can look at is the Klein-Gordon one:

$$\mathcal{L} = \frac{1}{2}\sum_i\left(\sum_\mu(\partial_\mu\eta_i^*)(\partial^\mu\eta_i) + m^2|\eta_i^2|\right)$$

which gives the Klein-Gordon equation

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2 - m^2\right)\eta_i = 0.$$

Today

Up to this point we have been considering symmetries which act on the entire configuration space uniformly, so the states of the system are transformed by one single group transformation.

$$\eta_i(x^\mu) \to \Gamma_{ij}(\mathfrak{g})\eta_j(x^\mu),$$

with the same group element $\mathfrak{g}$ at all space-time points. As we assume the representation Γ under which η transforms is unitary, $\mathcal{L}$ is invariant and we have a (global) symmetry. The global symmetry transformations we have considered, largely in classifying states of a system, are very useful, but not nearly as sexy as the idea of gauge symmetry groups, which we now understand as underlying all of fundamental physics. The basic idea is that as the physical degrees of freedom are local and interact locally, they ought to have symmetry properties independently at each space-time point, a local symmetry.

If we consider such a possibility,

$$\eta_i(x^\mu) \to \Gamma_{ij}\left(\mathfrak{g}(x^\mu)\right)\eta_j(x^\mu),$$

terms in $\mathcal{L}$ which depend only on η_i and not its derivatives will be invariant, but the derivative terms will have additional pieces involving the derivative of $\mathfrak{g}$. Of course without such terms the Lagrangian is not a field theory but an infinite collection of independent points, so we need to figure out how to give meaning to $\vec{\nabla}\eta$ which respects the position-dependent group transformation. We will see that this will require introducing new dynamical degrees of freedom, *gauge fields*.

In order to better understand the structure of a gauge field theory, we will consider how to get such a theory from the continuum limit of a lattice. The final result will not involve the lattice but it will have motivated the form of the piece of the lagrangian depending solely on the gauge fields. In the simplest case, this is the $\frac{1}{2}(\vec{E}^{\,2} - \vec{B}^{\,2})$ term in $\mathcal{L}$ for electromagnetism. But we will see that it is not a pure quadratic in the gauge field if the symmetry group is not Abelian, which means the gauge particles, unlike photons, will have interactions independent of the "matter" (or charged) fields.

Reminder: Eighth Homework due April 6 at 4:00 PM