Last Friday

We were looking to find all the finite-dimensional representations of the Lie groups, which we observed could be extracted from tensor products of fundamental representations $D^i$. As for SU(3) there are only two fundamental representations, and as the highest weight will come from symmetrizing all the $D^1$’s and all the $D^2$’s, we needed only extract pieces involving tracing one upper and one lower index to get the irreducible representation with $\vec{\mu}_{\text{max}} = q^1 \vec{\mu}^1 + q^2 \vec{\mu}^2$.

Today

For SU(N) with $N \geq 3$ there are more than two fundamental representations and the same approach is considerably more awkward. It turns out that there is another method, possible for SU(N) because all of the fundamental representations can in fact be found from antisymmetrizing tensor products of the defining representation. We will see that explicitly for SU(3), and Georgi proves it exhaustively for higher N. This will mean that we in fact only need tensors in the defining representation, but with complicated symmetrization. This leads us into the next chapter, on representations of the permutation group $S_k$.

We will define the group algebra of any finite group as a formal sum of the group elements multiplied by coefficients in a field, which for us is generally $\mathbb{C}$. I will define what an algebra\(^1\) is, even though you thought you learned algebra in high school or before. The elements of the group algebra don’t have an intuitive meaning by themselves, but they do satisfy the requirements of a noncommutative but associative algebra, with ordinary vector space addition and scalar multiplication, and multiplication given by the group multiplication law. While the elements of the algebra don’t have an intuitive interpretation, their action on functions on the group or on group

\(^1\)Definition: An algebra consists of a vector space $V$ over a field $F$, together with a binary operation of multiplication on the set $V$ of vectors, such that for all $a \in F$ and $\alpha, \beta, \gamma \in V$, the following are satisfied:

1. $(aa)\beta = a(a\beta) = a(\alpha\beta)$
2. $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$
3. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

$V$ is an associative algebra over $F$ if, in addition,

4. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in V$. 

1
modules (to use the math word) is intuitive. In particular, given a function \( f \) on \( k \) variables of the same type, \( \frac{1}{k!} \sum_P P f \) is the totally symmetric part of \( f \), and \( \frac{1}{k!} \sum_P (-1)^P P f \) is the totally antisymmetric part of \( f \), but extracting the other parts requires other elements of the \( S_k \) group algebra.

The elements of the \( S_k \) group algebra we are looking for will be found in terms of the irreducible representations of \( S_k \). There is one irreducible representation for each conjugacy class, and as you showed in homework #4, there is one for each partition of \( k \). We will define a Young graph as an upside-down tower of \( k \) square boxes, left justified, and then a Young tableau as a Young graph with \((1, ..., k)\) inserted in the boxes in some order. Each tableau \( \tau \) will have one element \( P_\tau \) of the group algebra associated with it as a symmetrizer, and another \( Q_\tau \) as an antisymmetrizer. Then the product \( Y_\tau = Q_\tau P_\tau \) is called the Young operator (and is, of course, an element in the group algebra), but actually we will be more interested in \( Q_\tau s_{\tau' \tau} P_\tau \) with \( \tau \) and \( \tau' \) a set of \textit{standard tableaux} corresponding to two Young tableau in the same Young graph, and \( s_{\tau' \tau} \) is the permutation connecting them. This is rather algebraically involved.

We will apply this arsenal of algebraic algorithms to explore the representations of \( SU(n) \).

By the way, an exhaustive treatment of this material is in an old self-published book by Schensted, still available, but for about 10 times as much as I paid. See the Book Refs web page.

Announcements:

The midterm exam performance was somewhat disappointing, although it is not unusual for my exams to have averages in the 40's, as did this one. I would like to say that

- If I assume you know something you don’t (like what addition or multiplication mod \( p \) means), please ask me, don’t wait until it costs you dearly on an exam.

- It is a good idea to answer the questions asked, in the places asked. That did not always happen.

I feel a bit guilty about not guiding you through the \( q - p \) box drawing methods of Georgi in constructing the full adjoint representation from the Cartan matrix. I have added a note to the Supplementary Notes web page which gives what I think is an improved diagrammatic prescription, in \textit{On the q-p diagrams of Georgi}. You will not be responsible for this material, but I include it for completeness.

Have a happy holiday!