Chapter 5

Semisimple Compact Lie Groups

We return now to considering a general finite dimensional semisimple compact Lie group and its Lie algebra.

In the algebra there are many abelian subalgebras, though not invariant. For example, any one dimensional subspace is an abelian subalgebra. If there are several generators which commute with each other, then one can form a larger abelian subspace. Let us take a maximal abelian subalgebra $H \subset \mathcal{L}$, of dimension m, which means there is no abelian subalgebra of larger dimension. H is called a **Cartan subalgebra**, with basis H_i. m is the **rank** of $\mathcal{L}$.

Consider a representation D of $\mathcal{L}$. The matrices corresponding to the H_i may all be simultaneously diagonalized, as they commute, so we label the basis vectors μ, we have

$$H_i \left|\mu D\right\rangle = \mu_i \left|\mu D\right\rangle .$$

The eigenvalues μ_i are called the **weights** and the vector $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_m)$ is called the **weight vector** corresponding to the basis vector μ. Notice this is a vector in a m-dimensional space, where m is the *rank*, not the dimension of the full Lie algebra $\mathcal{L}$ or of the representation.

There is a *vector* $\vec{\mu}$ for each dimension μ of the *representation*. But unlike for SU(2), there may be several basis vectors in the representation with the same $\vec{\mu}$, so we ought to include a subsidiary label x, $H_i \left|\mu, x, D\right\rangle = \mu_i \left|\mu, x, D\right\rangle$.

We consider in particular the adjoint representation. Then the basis vectors are also the Lie algebra generators L_a. From compact semisimplicity

we know we can choose $\operatorname{Tr} L_a L_b = \lambda \delta_{ab}$ for the real basis vectors[1]. The generators act on the states by[2]

$$L_a \left|L_b\right\rangle = \left|[L_a, L_b]\right\rangle = i f_{ab}{}^c \left|L_c\right\rangle .$$

We will consider states in the complexification of $\mathcal{L}$, so as to include things like $J_\pm$, so we can write in general

$$\langle X_1 | X_2 \rangle = \lambda^{-1} \operatorname{Tr}\left(X_1^\dagger X_2\right)$$

for X_1 and X_2 any elements of the complexification of $\mathcal{L}$, $X = \sum_a \nu_a L_a$ with $\nu_a \in \mathbb{C}$.

From above, $H_i \left|H_j\right\rangle = 0$. The rest of the vector space $\mathcal{L}$ has a basis which, having diagonalized H, satisfies

$$H_i \left|E_{\vec{\alpha}}\right\rangle = \alpha_i \left|E_{\vec{\alpha}}\right\rangle .$$

The H_i are hermitean and diagonal, but the eigenvectors $E_{\vec{\alpha}}$ are not necessarily real combinations of the hermitean L_i, so $E_{\vec{\alpha}}$ is not necessarily equal to $E^\dagger_{\vec{\alpha}}$. Nonetheless α_i is real, as all the eigenvalues of a hermitean matrix are, so, as

$$\begin{aligned} [H_i, E_{\vec{\alpha}}] &= \alpha_i E_{\vec{\alpha}}, \\ \left[H_i, E^\dagger_{\vec{\alpha}}\right] &= -\alpha_i E^\dagger_{\vec{\alpha}}. \end{aligned}$$

For each $\vec{\alpha}$, the α_i, $i = 1, \dots m$, are not all zero because we assumed $E_{\vec{\alpha}}$ is not in the Cartan subalgebra. So $E^\dagger_{\vec{\alpha}}$ is another eigenvector $E_{-\vec{\alpha}}$. Normalize the $E_{\vec{\alpha}}$'s so that $\langle E_{\vec{\alpha}}, E_{\vec{\beta}} \rangle = \delta_{\vec{\alpha},\vec{\beta}}$ as well as $\langle H_i, H_j \rangle = \delta_{ij}$.

The $\{\alpha_i\}$ for all the $E_{\vec{\alpha}}$ are the non-zero weights of the adjoint representation, and are called the **roots** of the algebra. Again, each root is in the m-dimensional vector space.

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[1] Slight change in notation: assuming we are using a basis with diagonalized $\beta_{ab} = \lambda \delta_{ab}$, we change the name of the structure constants $c_{ij}{}^k \to f_{ijk}$.

[2] The choice of basis vectors that makes the Killing form essentially the unit matrix enables us to show the f's are totally antisymmetric, for

$$\begin{aligned} i\lambda f_{ab}{}^c &= \operatorname{Tr}\left([L_a, L_b] L_c\right) = \operatorname{Tr}\left(L_a L_b L_c\right) - \operatorname{Tr}\left(L_b L_a L_c\right) = \operatorname{Tr}\left(L_a L_b L_c\right) - \operatorname{Tr}\left(L_a L_c L_b\right) \\ &= \operatorname{Tr}\left(L_a [L_b, L_c]\right) = i\lambda f_{bc}{}^a . \end{aligned}$$

The $E_{\vec\alpha}$ generators act as raising operators in some direction for any representation. If we have a state of weight $\vec\mu$ in representation D, (so $H_i\left|\mu D\right\rangle = \mu_i\left|\mu D\right\rangle$) then $E_{\vec\alpha}\left|\mu D\right\rangle$ is a state of weight $\vec\mu+\vec\alpha$, for

$$H_iE_{\vec\alpha}\left|\mu D\right\rangle = \underbrace{[H_i,E_{\vec\alpha}]}_{\alpha_iE_{\vec\alpha}}\left|\mu D\right\rangle + E_{\vec\alpha}\underbrace{H_i\left|\mu D\right\rangle}_{\mu_i\left|\mu D\right\rangle} = (\alpha_i+\mu_i)E_{\vec\alpha}\left|\mu D\right\rangle,$$

unless, of course, it vanishes with $E_{\vec\alpha}\left|\mu D\right\rangle = 0$.

In the adjoint representation consider the state $E_{\vec\alpha}\left|E_{-\vec\alpha}\right\rangle$. It has weight 0, so commutes with each H_i and must be contained in the Cartan subalgebra, for otherwise it could have been added to it. Thus $E_{\vec\alpha}\left|E_{-\vec\alpha}\right\rangle = \sum_i\beta_i\left|H_i\right\rangle$, or $[E_{\vec\alpha},E_{-\vec\alpha}] = \sum_i\beta_iH_i$. As $[E_{\vec\alpha},E_{-\vec\alpha}]$ is hermitian, β_i is real. Note

$$\beta_i = \left\langle H_i\right| E_{\vec\alpha}\left|E_{-\vec\alpha}\right\rangle = \left\langle E_{-\vec\alpha}\right| E_{-\vec\alpha}\left|H_i\right\rangle^* = \left\langle E_{-\vec\alpha}\right|[E_{-\vec\alpha},H_i]\rangle^* = \alpha_i^* = \alpha_i,$$

so

$$[E_{\vec\alpha},E_{-\vec\alpha}] = \sum_i\alpha_iH_i.$$

Now consider an arbitrary finite dimensional representation D and a state of definite weight $\vec\mu$. Just as we did with L_+ in SU(2), we will generate states with $E_{\pm\vec\alpha}$ until they vanish. Consider a specific $E_{\vec\alpha}$, and apply $E_{\vec\alpha}$ until you get to a state $\left|\vec\mu{}'D\right\rangle$ which vanishes upon further application of $E_{\vec\alpha}$, so $E_{\vec\alpha}\left|\vec\mu{}'D\right\rangle = 0$, with $\vec\mu{}' = \vec\mu+p\vec\alpha$ for some integer $p\geq 0$. Normalize the state $\left|\vec\mu{}'D\right\rangle$, and generate therefrom a sequence of normalized states

$$\left|\vec\mu{}'-n\vec\alpha,D\right\rangle = N_n^{-1}E_{-\vec\alpha}\left|\vec\mu{}'-(n-1)\vec\alpha,D\right\rangle$$

as long as $E_{-\vec\alpha}$ can be applied without killing the state. The N_n is a real positive normalization factor so that all the states are normalized.

Thus

$$\left|\vec\mu{}'-n\vec\alpha,D\right\rangle = \left(\prod_{r=1}^n N_r^{-1}\right)E_{-\vec\alpha}^n\left|\vec\mu{}',D\right\rangle,$$

$$\begin{aligned}
E_{\vec\alpha}\left|\vec\mu{}'-n\vec\alpha,D\right\rangle &= \left(\prod_{r=1}^n N_r^{-1}\right)\sum_{r=0}^{n-1}E_{-\vec\alpha}^{n-r-1}\left[E_{\vec\alpha},E_{-\vec\alpha}\right]E_{-\vec\alpha}^r\left|\vec\mu{}',D\right\rangle\\
&= \left(\prod_{r=1}^n N_r^{-1}\right)\sum_{r=0}^{n-1}E_{-\vec\alpha}^{n-r-1}\underbrace{\sum_i\alpha_iH_i\,E_{-\vec\alpha}^r\left|\vec\mu{}',D\right\rangle}_{(\vec\mu{}'-r\vec\alpha)\cdot\vec\alpha\,E_{-\vec\alpha}^r\left|\vec\mu{}',D\right\rangle}\\
&= N_n^{-1}\left(n(\vec\mu{}'\cdot\vec\alpha)-\frac{1}{2}n(n-1)\alpha^2\right)\underbrace{\left(\prod_{r=1}^{n-1}N_r^{-1}\right)E_{-\vec\alpha}^{n-1}\left|\vec\mu{}',D\right\rangle}_{\left|\vec\mu-(n-1)\vec\alpha,D\right\rangle}.
\end{aligned}$$

So

$$\begin{aligned}
\left\langle\vec\mu-(n-1)\vec\alpha,D\right|E_{\vec\alpha}\left|\vec\mu{}'-n\vec\alpha,D\right\rangle &= \frac{n(\vec\mu{}'\cdot\vec\alpha)-\frac12 n(n-1)\alpha^2}{N_n}\\
=\left\langle\vec\mu{}'-n\vec\alpha,D\right|E_{-\vec\alpha}\left|\vec\mu-(n-1)\vec\alpha,D\right\rangle^* &= N_n,
\end{aligned}$$

so $|N_n|^2 = n\vec\mu{}'\cdot\vec\alpha-\frac12 n(n-1)\alpha^2$.

For a finite dimensional representation, $N_{n+1}=0$ for some $n\geq 0$, so

$$(n+1)\left[\vec\mu{}'\cdot\vec\alpha-\frac12 n\alpha^2\right]=0,\qquad\text{or}\quad\frac{\vec\mu{}'\cdot\vec\alpha}{\alpha^2}=\frac{n}{2}.$$

For the original vector $\vec\mu$,

$$\frac{\vec\mu\cdot\vec\alpha}{\alpha^2} = \frac{(\vec\mu{}'-p\vec\alpha)\cdot\vec\alpha}{\alpha^2} = \frac{n}{2}-p.$$

If we had applied the lowering operator $E_{-\vec\alpha}$ to $\left|\vec\mu,D\right\rangle$ directly, we would have gotten zero after some integer number $q+1$ of applications of $E_{-\vec\alpha}$, with $n+1=q+1+p$, so we may write

$$\frac{\vec\mu\cdot\vec\alpha}{\alpha^2} = \frac{q-p}{2}.$$

In the adjoint representation $\vec\mu$ is a root, say $\vec\beta$, so $\dfrac{\vec\alpha\cdot\vec\beta}{\alpha^2}=\dfrac{n}{2}$ for every pair of roots $\vec\alpha$ and $\vec\beta$, for some integer n. Multiplying by the same relation with $\vec\alpha\leftrightarrow\vec\beta$,

$$\frac{(\vec\alpha\cdot\vec\beta)^2}{\alpha^2\beta^2} = \cos^2\theta = \frac{m}{4},$$

with m an integer which must be 0, 1, 2, 3, or 4. θ is the angle between the two roots. Thus the possible angles between roots are 0, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°.

Of course every root has an angle of 0° with itself and 180° with its conjugate. We now show that the angle between two different generators E_α and E_β cannot be 0. For suppose $\vec\alpha \parallel \vec\beta$ with $|\vec\beta| \geq |\vec\alpha|$. Then as

$$\frac{\vec\alpha\cdot\vec\beta}{\beta^2} = \frac{|\vec\alpha|}{|\vec\beta|} \leq 1,$$

the only possibilities are $\vec\beta = 2\vec\alpha$ or $\vec\beta = \vec\alpha$. First consider the possibility that $\vec\alpha = \vec\beta$, that two different vectors $E_{\vec\alpha}$ and $E'_{\vec\alpha}$ have the same root. They can be chosen orthogonal. Now $E_{-\vec\alpha}\,|E'_{\vec\alpha}\rangle = \sum_i \beta_i H_i$ as it has weight zero. Then

$$\begin{aligned}\beta_i &= \langle H_i|\,E_{-\vec\alpha}\,|E'_{\vec\alpha}\rangle = \langle E'_{\vec\alpha}|\,E_{\vec\alpha}\,|H_i\rangle^* = \langle E'_{\vec\alpha}|[E_{\vec\alpha}, H_i]\rangle^* = -\langle E'_{\vec\alpha}|[H_i, E_{\vec\alpha}]\rangle^* \\ &= -\langle E'_{\vec\alpha}|\,H_i\,|E_{\vec\alpha}\rangle^* = -\alpha_i\langle E'_{\vec\alpha}|E_{\vec\alpha}\rangle^* = 0,\end{aligned}$$

so the number of times $E_{-\vec\alpha}$ can lower $|E'_{\vec\alpha}\rangle$ is zero, $q = 0$, and $\frac{\vec\alpha\cdot\vec\alpha}{\alpha^2} = 1 = \frac{q-p}{2} = -p/2$, which is impossible as $p \geq 0$. Therefore no root has two independent eigenvectors.

Now consider $\vec\beta = 2\vec\alpha$. $\vec\alpha + \vec\beta$ can't be a root, as it would be $3\vec\alpha$, so $E_{\vec\alpha}$ cannot raise $\left|E_{\vec\beta}\right\rangle$, the relevant $p = 0$, but $n = 4$, so $E_{-\vec\alpha}\left|E_{\vec\beta}\right\rangle \neq 0$, and it has weight $\vec\alpha$, so must be proportional to $|E_{\vec\alpha}\rangle$, $E_{-\vec\alpha}\left|E_{\vec\beta}\right\rangle = k\,|E_{\vec\alpha}\rangle$. Then $k = \langle E_{\vec\alpha}|\,E_{-\vec\alpha}\left|E_{\vec\beta}\right\rangle = \left\langle E_{\vec\beta}\right|E_{\vec\alpha}\,|E_{\vec\alpha}\rangle^* = \left\langle E_{\vec\beta}\right|[E_{\vec\alpha}, E_{\vec\alpha}]\rangle^* = 0$, so again we have a contradiction, which rules $\theta \neq 0$ for different roots, and $\theta \neq 180°$ for non-conjugate roots.

Notice that $E_{\pm\vec\alpha}$ and $\sum \alpha_i H_i$ form an SU(2) subalgebra, and the string of n values generated about a given $\vec\mu$ is a $j = n/2$ representation of this SU(2), and the above investigation into the possible values is a rehash of what we did for SU(2).

Before we go on, we need another example, and a very important one in elementary particle physics, SU(3).

But first a recap:

We are in the process of classifying all finite dimensional compact semisimple Lie algebras and their finite-dimensional irreducible representations.

$\mathcal{H}$ is the Cartan subalgebra of the Lie algebra $\mathcal{L}$ we are considering, with generators H_i for $\mathcal{H}$ and $E_{\vec\alpha}$ for the rest of $\mathcal{L}$.

$$\begin{aligned} H_i\,|\vec\mu, D\rangle &= \mu_i\,|\vec\mu, D\rangle\,, \quad \left\{\begin{array}{r}\vec\mu \text{ is the } \textbf{weight} \text{ of the state } |\vec\mu, D\rangle \\ \text{in representation } D\end{array}\right. \\ H_i\,|E_{\vec\alpha}\rangle &= \alpha_i\,|E_{\vec\alpha}\rangle\,, \quad \left\{\begin{array}{r}\vec\alpha \text{ is the } \textbf{root}\text{, the weight of a state in the} \\ \text{adjoint representation}\end{array}\right.\end{aligned}$$

Considering $E_{\vec\alpha}$ as a raising operator acting on a state $|\vec\mu, D\rangle$,
$p =$ number of times $E_{\vec\alpha}$ can act on $|\vec\mu, D\rangle$ before vanishing,
$q =$ number of times $E_{-\vec\alpha}$ can act on $|\vec\mu, D\rangle$ before vanishing.
Then $q - p = 2\,\dfrac{\vec\mu\cdot\vec\alpha}{\alpha^2}$.

If $\vec\alpha$ and $\vec\beta$ are roots, the angle θ between them has $4\cos^2\theta$ an integer, and $\theta \neq 0$ for different roots.