Chapter 5

Semisimple Compact Lie Groups

We return now to considering a general finite dimensional semisimple compact Lie group and its Lie algebra.

In the algebra there are many abelian subalgebras, though not invariant. For example, any one dimensional subspace is an abelian subalgebra. If there are several generators which commute with each other, then one can form a larger abelian subspace. Let us take a maximal abelian subalgebra $H \subset \mathcal{L}$, of dimension $m$, which means there is no abelian subalgebra of larger dimension. $H$ is called a Cartan subalgebra, with basis $H_i$. $m$ is the rank of $\mathcal{L}$.

Consider a representation $D$ of $\mathcal{L}$. The matrices corresponding to the $H_i$ may all be simultaneously diagonalized, as they commute, so we label the basis vectors $\mu$, we have

$$H_i |\mu D\rangle = \mu_i |\mu D\rangle .$$

The eigenvalues $\mu_i$ are called the weights and the vector $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$ is called the weight vector corresponding to the basis vector $\mu$. Notice this is a vector in a $m$-dimensional space, where $m$ is the rank, not the dimension of the full Lie algebra $\mathcal{L}$ or of the representation.

There is a vector $\vec{\mu}$ for each dimension $\mu$ of the representation. But unlike for SU(2), there may be several basis vectors in the representation with the same $\vec{\mu}$, so we ought to include a subsidiary label $x$, $H_i |\mu, x, D\rangle = \mu_i |\mu, x, D\rangle$.

We consider in particular the adjoint representation. Then the basis vectors are also the Lie algebra generators $L_a$. From compact semisimplicity
we know we can choose $\text{Tr} \, L_a L_b = \lambda \delta_{ab}$ for the real basis vectors\(^1\). The generators act on the states by\(^2\)

$$L_a |L_b\rangle = [[L_a, L_b]] = i f_{ab}^c |L_c\rangle.$$ 

We will consider states in the complexification of $\mathcal{L}$, so as to include things like $J_\pm$, so we can write in general

$$\langle X_1|X_2\rangle = \lambda^{-1} \text{Tr} \left( X_1^\dagger X_2 \right)$$

for $X_1$ and $X_2$ any elements of the complexification of $\mathcal{L}$, $X = \sum_a \nu_a L_a$ with $\nu_a \in \mathbb{C}$.

From above, $H_i |H_j\rangle = 0$. The rest of the vector space $\mathcal{L}$ has a basis which, having diagonalized $H$, satisfies

$$H_i |E_{\vec{\alpha}}\rangle = \alpha_i |E_{\vec{\alpha}}\rangle.$$ 

The $H_i$ are hermitean and diagonal, but the eigenvectors $E_{\vec{\alpha}}$ are not necessarily real combinations of the hermitean $L_i$, so $E_{\vec{\alpha}}$ is not necessarily equal to $E_{\vec{\alpha}}^\dagger$. Nonetheless $\alpha_i$ is real, as all the eigenvalues of a hermitean matrix are, so, as

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}},$$

$$[H_i, E_{\vec{\alpha}}^\dagger] = -\alpha_i E_{\vec{\alpha}}^\dagger.$$ 

For each $\vec{\alpha}$, the $\alpha_i$, $i = 1, \ldots, m$, are not all zero because we assumed $E_{\vec{\alpha}}$ is not in the Cartan subalgebra. So $E_{\vec{\alpha}}^\dagger$ is another eigenvector $E_{-\vec{\alpha}}$. Normalize the $E_{\vec{\alpha}}$'s so that $\langle E_{\vec{\alpha}}^\dagger, E_{\vec{\beta}}\rangle = \delta_{\vec{\alpha},\vec{\beta}}$ as well as $\langle H_i, H_j \rangle = \delta_{ij}$.

The $\{\alpha_i\}$ for all the $E_{\vec{\alpha}}$ are the non-zero weights of the adjoint representation, and are called the roots of the algebra. Again, each root is in the $m$-dimensional vector space.

\(^1\)Slight change in notation: assuming we are using a basis with diagonalized $\beta_{ab} = \lambda \delta_{ab}$, we change the name of the structure constants $c_{ij}^k \rightarrow f_{ijk}$.

\(^2\)The choice of basis vectors that makes the Killing form essentially the unit matrix enables us to show the $f$'s are totally antisymmetric, for

$$i \lambda f_{ab}^c = \text{Tr} \left( [L_a, L_b] L_c \right) = \text{Tr} \left( L_a L_b L_c \right) - \text{Tr} \left( L_b L_a L_c \right) = \text{Tr} \left( L_c L_a L_b \right) - \text{Tr} \left( L_a L_c L_b \right) = \text{Tr} \left( L_a [L_b, L_c] \right) = i \lambda f_{bc}^a.$$
The $E_{\bar{\alpha}}$ generators act as raising operators in some direction for any representation. If we have a state of weight $\vec{\mu}$ in representation $D$, (so $H_i |\mu D\rangle = \mu_i |\mu D\rangle$) then $E_{\bar{\alpha}} |\mu D\rangle$ is a state of weight $\vec{\mu} + \vec{\alpha}$, for

$$H_i E_{\bar{\alpha}} |\mu D\rangle = \left[H_i, E_{\bar{\alpha}}\right] |\mu D\rangle + E_{\bar{\alpha}} H_i |\mu D\rangle = (\alpha_i + \mu_i) E_{\bar{\alpha}} |\mu D\rangle ,$$

unless, of course, it vanishes with $E_{\bar{\alpha}} |\mu D\rangle = 0$.

In the adjoint representation consider the state $E_{\bar{\alpha}} |E_{-\bar{\alpha}}\rangle$. It has weight 0, so commutes with each $H_i$ and must be contained in the Cartan subalgebra, for otherwise it could have been added to it. Thus $E_{\bar{\alpha}} |E_{-\bar{\alpha}}\rangle = \sum_i \beta_i |H_i\rangle$, or $[E_{\bar{\alpha}}, E_{-\bar{\alpha}}] = \sum_i \beta_i H_i$. As $[E_{\bar{\alpha}}, E_{-\bar{\alpha}}]$ is hermitian, $\beta_i$ is real. Note

$$\beta_i = \langle H_i | E_{\bar{\alpha}} |E_{-\bar{\alpha}}\rangle = \langle E_{-\bar{\alpha}} | E_{-\bar{\alpha}} |H_i\rangle^* = \langle E_{-\bar{\alpha}} | [E_{-\bar{\alpha}}, H_i] \rangle^* = \alpha_i^* = \alpha_i,$$

so

$$[E_{\bar{\alpha}}, E_{-\bar{\alpha}}] = \sum_i \alpha_i H_i.$$

Now consider an arbitrary finite dimensional representation $D$ and a state of definite weight $\vec{\mu}$. Just as we did with $L_+$ in SU(2), we will generate states with $E_{\pm \bar{\alpha}}$ until they vanish. Consider a specific $E_{\bar{\alpha}}$, and apply $E_{\bar{\alpha}}$ until you get to a state $|\vec{\mu}'D\rangle$ which vanishes upon further application of $E_{\bar{\alpha}}$, so $E_{\bar{\alpha}} |\vec{\mu}'D\rangle = 0$, with $\vec{\mu}' = \vec{\mu} + p\vec{\alpha}$ for some integer $p \geq 0$. Normalize the state $|\vec{\mu}'D\rangle$, and generate therefrom a sequence of normalized states

$$|\vec{\mu}' - n\vec{\alpha}, D\rangle = N_n^{-1} E_{-\bar{\alpha}} |\vec{\mu}' - (n - 1)\vec{\alpha}, D\rangle$$

as long as $E_{-\bar{\alpha}}$ can be applied without killing the state. The $N_n$ is a real positive normalization factor so that all the states are normalized.

Thus

$$|\vec{\mu}' - n\vec{\alpha}, D\rangle = \left( \prod_{p=1}^{n} N_p^{-1} \right) E_{-\bar{\alpha}}^n |\vec{\mu}', D\rangle ,$$
\[ E_{\vec{\alpha}} | \vec{\mu}' - n\vec{\alpha}, D \rangle = \left( \prod_{r=1}^{n} N_{r}^{-1} \right) \sum_{r=0}^{n-1} E_{-\vec{\alpha}}^{-r-1} [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] E_{-\vec{\alpha}}^{r} | \vec{\mu}', D \rangle \]
\[
= \left( \prod_{r=1}^{n} N_{r}^{-1} \right) \sum_{r=0}^{n-1} E_{-\vec{\alpha}}^{-r-1} \sum_{i} \alpha_{i} H_{i} E_{-\vec{\alpha}}^{r} | \vec{\mu}', D \rangle \underbrace{ \langle \vec{\mu}' - r\vec{\alpha} | \vec{\mu}' \rangle \delta_{E_{-\vec{\alpha}}^{r} | \vec{\mu}', D} \rangle }_{\text{same relation}} \]
\[
= N_{n}^{-1} \left( n(\vec{\mu}' \cdot \vec{\alpha}) - \frac{1}{2} n(n-1)\alpha^{2} \right) \left( \prod_{r=1}^{n-1} N_{r}^{-1} \right) E_{-\vec{\alpha}}^{n-1} | \vec{\mu}', D \rangle \, .
\]

So
\[
\langle \vec{\mu} - (n-1)\vec{\alpha}, D | E_{\vec{\alpha}} | \vec{\mu}' - n\vec{\alpha}, D \rangle = \frac{n(\vec{\mu}' \cdot \vec{\alpha}) - \frac{1}{2} n(n-1)\alpha^{2}}{N_{n}} \]
\[
= \langle \vec{\mu}' - n\vec{\alpha}, D | E_{-\vec{\alpha}} | \vec{\mu} - (n-1)\vec{\alpha}, D \rangle = N_{n},
\]
so \(|N_{n}|^{2} = n\vec{\mu}' \cdot \vec{\alpha} - \frac{1}{2} n(n-1)\alpha^{2} \).

For a finite dimensional representation, \(N_{n+1} = 0\) for some \(n \geq 0\), so
\[
(n + 1) \left[ \vec{\mu}' \cdot \vec{\alpha} - \frac{1}{2} n\alpha^{2} \right] = 0, \quad \text{or} \quad \frac{\vec{\mu}' \cdot \vec{\alpha}}{\alpha^{2}} = \frac{n}{2}.
\]

For the original vector \(\vec{\mu}\),
\[
\frac{\vec{\mu} \cdot \vec{\alpha}}{\alpha^{2}} = \frac{(\vec{\mu}' - p\vec{\alpha}) \cdot \vec{\alpha}}{\alpha^{2}} = \frac{n}{2} - p.
\]

If we had applied the lowering operator \(E_{-\vec{\alpha}}\) to \(|\vec{\mu}, D\rangle\) directly, we would have gotten zero after some integer number \(q+1\) of applications of \(E_{-\vec{\alpha}}\), with \(n+1 = q+1+p\), so we may write
\[
\frac{\vec{\mu} \cdot \vec{\alpha}}{\alpha^{2}} = \frac{q-p}{2}.
\]

In the adjoint representation \(\vec{\mu}\) is a root, say \(\vec{\beta}\), so \(\vec{\alpha} \cdot \vec{\beta} = \frac{n}{2}\) for every pair of roots \(\vec{\alpha}\) and \(\vec{\beta}\), for some integer \(n\). Multiplying by the same relation with \(\vec{\alpha} \leftrightarrow \vec{\beta}\),
\[
\frac{(\vec{\alpha} \cdot \vec{\beta})^{2}}{\alpha^{2}\beta^{2}} = \cos^{2} \theta = \frac{m}{4},
\]

For a finite dimensional representation, \(N_{n+1} = 0\) for some \(n \geq 0\), so
\[
(n + 1) \left[ \vec{\mu}' \cdot \vec{\alpha} - \frac{1}{2} n\alpha^{2} \right] = 0, \quad \text{or} \quad \frac{\vec{\mu}' \cdot \vec{\alpha}}{\alpha^{2}} = \frac{n}{2}.
\]

For the original vector \(\vec{\mu}\),
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\frac{\vec{\mu} \cdot \vec{\alpha}}{\alpha^{2}} = \frac{(\vec{\mu}' - p\vec{\alpha}) \cdot \vec{\alpha}}{\alpha^{2}} = \frac{n}{2} - p.
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\[
\frac{(\vec{\alpha} \cdot \vec{\beta})^{2}}{\alpha^{2}\beta^{2}} = \cos^{2} \theta = \frac{m}{4},
\]
with \( m \) an integer which must be 0, 1, 2, 3, or 4. \( \theta \) is the angle between the two roots. Thus the possible angles between roots are 0, 30\(^\circ\), 45\(^\circ\), 60\(^\circ\), 90\(^\circ\), 120\(^\circ\), 135\(^\circ\), 150\(^\circ\), 180\(^\circ\).

Of course every root has an angle of 0\(^\circ\) with itself and 180\(^\circ\) with its conjugate. We now show that the angle between two different generators \( E_\alpha \) and \( E_\beta \) cannot be 0. For suppose \( \vec{\alpha} \parallel \vec{\beta} \) with \( |\vec{\beta}| \geq |\vec{\alpha}| \). Then as

\[
\frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\beta}|^2} = \frac{|\vec{\alpha}|}{|\vec{\beta}|} \leq 1,
\]

the only possibilities are \( \vec{\beta} = 2\vec{\alpha} \) or \( \vec{\beta} = \vec{\alpha} \). First consider the possibility that \( \vec{\alpha} = \vec{\beta} \), that two different vectors \( E_\alpha \) and \( E_\beta \) have the same root. They can be chosen orthogonal. Now \( E_{-\alpha}|E'_\beta\rangle = \sum_i \beta_i H_i \) as it has weight zero. Then

\[
\beta_i = \langle H_i | E_{-\alpha}|E'_\beta\rangle = \langle E'_\beta| E_\alpha | H_i \rangle^* = \langle E'_\beta|[E_\alpha, H_i]|E_\alpha \rangle^* = -\langle E'_\beta|[H_i, E_\alpha]|E_\alpha \rangle^* = -\langle \sum_i \alpha_i E'_\beta| E_\alpha \rangle^* = 0,
\]

so the number of times \( E_{-\alpha} \) can lower \( |E'_\beta\rangle \) is zero, \( q = 0 \), and \( \frac{q-p}{2} = q = -p/2 \), which is impossible as \( p \geq 0 \). Therefore no root has two independent eigenvectors.

Now consider \( \vec{\beta} = 2\vec{\alpha} \). \( \vec{\alpha} + \vec{\beta} \) can’t be a root, as it would be \( 3\vec{\alpha} \), so \( E_\alpha \) cannot raise \( |E_{-\beta}\rangle \), the relevant \( p = 0 \), but \( n = 4 \), so \( E_{-\alpha}|E_{-\beta}\rangle \neq 0 \), and it has weight \( \vec{\alpha} \), so must be proportional to \( |E_{\alpha}\rangle, E_{-\alpha}|E_{-\beta}\rangle = k|E_{\alpha}\rangle \). Then

\[
k = \langle E_{-\alpha}|E_{-\alpha}|E_{-\beta}\rangle = \langle E_{-\beta}|E_{-\alpha}|E_{-\beta}\rangle^* = \langle E_{-\beta}|E_{-\beta}|E_{\alpha}\rangle = 0,
\]

so again we have a contradiction, which rules \( \theta \neq 0 \) for different roots, and \( \theta \neq 180^\circ \) for non-conjugate roots.

Notice that \( E_{\pm \alpha} \) and \( \sum \alpha_i H_i \) form an SU(2) subalgebra, and the string of \( n \) values generated about a given \( \vec{\mu} \) is a \( j = n/2 \) representation of this SU(2), and the above investigation into the possible values is a rehash of what we did for SU(2).

Before we go on, we need another example, and a very important one in elementary particle physics, SU(3).

But first a recap:

We are in the process of classifying all finite dimensional compact semisimple Lie algebras and their finite-dimensional irreducible representations.
$\mathcal{H}$ is the Cartan subalgebra of the Lie algebra $\mathcal{L}$ we are considering, with generators $H_i$ for $\mathcal{H}$ and $E_{\vec{\alpha}}$ for the rest of $\mathcal{L}$.

$$H_i |\vec{\mu}, D\rangle = \mu_i |\vec{\mu}, D\rangle, \quad \{ \vec{\mu} \text{ is the weight of the state } |\vec{\mu}, D\rangle \text{ in representation } D \}$$

$$H_i |E_{\vec{\alpha}}\rangle = \alpha_i |E_{\vec{\alpha}}\rangle, \quad \{ \vec{\alpha} \text{ is the root, the weight of a state in the adjoint representation} \}$$

Considering $E_{\vec{\alpha}}$ as a raising operator acting on a state $|\vec{\mu}, D\rangle$,

$p = \text{number of times } E_{\vec{\alpha}} \text{ can act on } |\vec{\mu}, D\rangle \text{ before vanishing,}$

$q = \text{number of times } E_{-\vec{\alpha}} \text{ can act on } |\vec{\mu}, D\rangle \text{ before vanishing.}$

Then $q - p = 2 \frac{\vec{\mu} \cdot \vec{\alpha}}{\alpha^2}$.

If $\vec{\alpha}$ and $\vec{\beta}$ are roots, the angle $\theta$ between them has $4 \cos^2 \theta$ an integer, and $\theta \neq 0$ for different roots.